Control Scheduling Subject to Matroid Constraints

Luiz F. O. Chamon,
Alexandre Amice, and Alejandro Ribeiro

CDC 2019
December 12th, 2019

Why?

Size	**Heterogeneity**

Why?

Size	**Heterogeneity**
Underactuation	

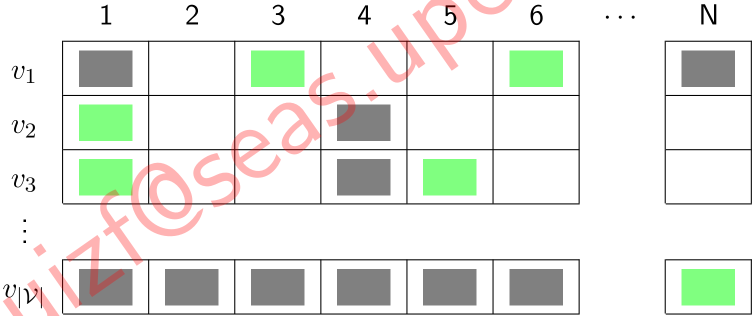
Why?

Size	Heterogeneity
Underactuation	Scheduling constraints

Goal

Problem (Control scheduling)

Assign inputs to time slots to minimize a control cost under budget and operational constraints.



The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- $\mathcal{V}_k = \{v_1^k, v_2^k, \dots\}$ is the set of inputs available at time k
- $\overline{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_N$

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- $\mathcal{V}_k = \{v_1^k, v_2^k, \dots\}$ is the set of inputs available at time k
- $\overline{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_N$
- $\mathcal{C} \subset 2^{\overline{\mathcal{V}}}$ is a collection of valid schedules

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- $\mathcal{V}_k = \{v_1^k, v_2^k, \dots\}$ is the set of inputs available at time k
- $\overline{\mathcal{V}} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \dots \cup \mathcal{V}_N$
- $\mathcal{C} \subset 2^{\overline{\mathcal{V}}}$ is a collection of valid schedules
- f is the control cost (LQR)

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \sum_{i \in \mathcal{S} \cap \mathcal{V}_k} \boldsymbol{b}_i u_{i,k}$$

$$\boldsymbol{x}_0 \sim \mathcal{N}(\bar{\boldsymbol{x}}_0, \boldsymbol{\Pi}_0)$$

Problem (LQR)

$$f(\mathcal{S}) = \min_{\mathcal{U}(\mathcal{S})} \mathbb{E}\left[\sum_{k=0}^{N-1}\left(\boldsymbol{x}_k^T\boldsymbol{Q}\boldsymbol{x}_k + \sum_{i \in \mathcal{S} \cap \mathcal{V}_k} r_i u_{i,k}^2\right) + \boldsymbol{x}_N^T\boldsymbol{Q}\boldsymbol{x}_N\right]$$

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \sum_{i \in \mathcal{S} \cap \mathcal{V}_k} \boldsymbol{b}_i u_{i,k}$$

$$\boldsymbol{x}_0 \sim \mathcal{N}(\bar{\boldsymbol{x}}_0, \boldsymbol{\Pi}_0)$$

Solution (LQR)

$$f(\mathcal{S}) = \operatorname{Tr}[\boldsymbol{\Sigma}_0 \boldsymbol{P}_0(\mathcal{S})]$$

$$\boldsymbol{P}_N(\mathcal{S}) = \boldsymbol{Q}$$

$$\boldsymbol{P}_k(\mathcal{S}) = \boldsymbol{Q} + \boldsymbol{A}^T \left(\boldsymbol{P}_{k+1}^{-1}(\mathcal{S}) + \sum_{i \in \mathcal{S} \cap \mathcal{V}_k} r_i^{-1} \boldsymbol{b}_i \boldsymbol{b}_i^T \right)^{-1} \boldsymbol{A}$$

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- Complexity depends on the anatomy of $\mathcal{C}$ and f
- For arbitrary $\mathcal{C}$, finding a feasible schedule can be hard
- Even for simple $\mathcal{C}$ ($|\mathcal{S}| \leq s$), control scheduling is NP-hard
  [Natarajan'95, Zhang'17, Ye'17]

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?

2. How close can we get to the optimal schedule?

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?
   Matroids
2. How close can we get to the optimal schedule?
   α-supermodularity

What scheduling constraints are tractable?

How close can we get to the optimal schedule?

When is greedy search good enough?

How good is greedy LQR scheduling?

Scheduling constraints

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- Unconstrained problem is trivial

- Arbitrary $\mathcal{C}$: finding a feasible schedule may be as hard as finding the optimal one

Scheduling constraints

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- Unconstrained problem is trivial
- Matroids
- Arbitrary $\mathcal{C}$: finding a feasible schedule may be as hard as finding the optimal one

- Extend the concept of linear independence to arbitrary algebraic structures

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Matroids

Definition

A *matroid* $M = (\overline{\mathcal{V}}, \mathcal{I})$ consists of a finite set of elements $\mathcal{E}$ and a family $\mathcal{I} \subseteq 2^{\overline{\mathcal{V}}}$ of subsets of $\overline{\mathcal{V}}$ called *independent sets* that satisfy:

1. $\emptyset \in \mathcal{I}$;
2. if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \in \mathcal{I}$, then $\mathcal{A} \in \mathcal{I}$;
3. if $\mathcal{A}, \mathcal{B} \in \mathcal{I}$ and $|\mathcal{A}| < |\mathcal{B}|$, then there exists $u \in \mathcal{B} \setminus \mathcal{A}$ such that $\mathcal{A} \cup \{u\} \in \mathcal{I}$.

What's in a matroid?

- Bound on the total number of control actions:
  $\mathcal{I} = \{\mathcal{S} \subseteq \overline{\mathcal{V}} \mid |\mathcal{S}| \leq s\}$
- Bound on the number of inputs used per time slot:
  $\mathcal{I} = \{\mathcal{S} \subseteq \overline{\mathcal{V}} \mid |\mathcal{S} \cap \mathcal{V}_k| \leq s_k\}$
- Bound on the number of times an input is used:
  $\mathcal{I} = \{\mathcal{S} \subseteq \overline{\mathcal{V}} \mid \left|\mathcal{S} \cap \{v_j^1, \ldots, v_j^N\}\right| \leq s_j\}$
- Restriction on the consecutive use of inputs:
  $\mathcal{I} = \{\mathcal{S} \subseteq \overline{\mathcal{V}} \mid v_j^k \notin \mathcal{S} \text{ or } v_j^{k+1} \notin \mathcal{S}\}$

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

$$\mathcal{C} = \bigcap_{p=1}^{P} \mathcal{I}_p \text{ such that } (\overline{\mathcal{V}}, \mathcal{I}_p) \text{ are matroids}$$

- Not necessarily a matroid...
- ...but preserves the important structures

Matroid intersection

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

$$\mathcal{C} = \bigcap_{p=1}^{P} \mathcal{I}_p \text{ such that } (\overline{\mathcal{V}}, \mathcal{I}_p) \text{ are matroids}$$

- Not necessarily a matroid...
- ...but preserves the important structures
  - Inheritance: schedules in $\mathcal{C}$ can be built element-by-element

Definition

Select inputs one at a time by choosing one yields a feasible schedule

$$\mathcal{G}_0 = \emptyset$$

for $j = 1, 2, \ldots$

$$g \in \overline{\mathcal{V}}$$

$$\text{subject to} \quad \mathcal{G}_{j-1} \cup \{g\} \in \mathcal{C}$$

$$\mathcal{G}_j = \mathcal{G}_{j-1} \cup \{g\}$$

end

Definition

Select inputs one at a time by choosing one yields a feasible schedule and (locally) minimizes the objective

$$\mathcal{G}_0 = \emptyset$$

for $j = 1, 2, \ldots$

$$g \in \underset{v \in \overline{\mathcal{V}} \setminus \mathcal{G}_{j-1}}{\operatorname{argmin}} \quad f\left(\mathcal{G}_{j-1} \cup \{v\}\right)$$

$$\text{subject to} \quad \mathcal{G}_{j-1} \cup \{v\} \in \mathcal{C}$$

$$\mathcal{G}_j = \mathcal{G}_{j-1} \cup \{g\}$$

end

Matroid intersection

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

$$\mathcal{C} = \bigcap_{p=1}^{P} \mathcal{I}_p \text{ such that } (\overline{\mathcal{V}}, \mathcal{I}_p) \text{ are matroids}$$

- Not necessarily a matroid...
- ...but preserves the important structures
  - Inheritance: schedules in $\mathcal{C}$ can be built element-by-element
  - Partial augmentation: greedy schedules are *large enough*

What scheduling constraints are tractable?

How close can we get to the optimal schedule?

When is greedy search good enough?

How good is greedy LQR scheduling?

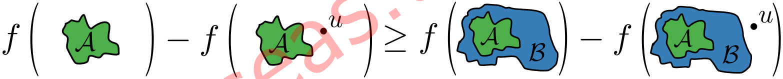
$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- $1/(1+P)$-optimal when f is supermodular
  [Fisher'78, Conforti'84]

Definition (Supermodularity)

For $\mathcal{A} \subset \mathcal{B} \subseteq \overline{\mathcal{V}}$ and $u \in \overline{\mathcal{V}} \setminus \mathcal{B}$

$$f(\mathcal{A}) - f(\mathcal{A} \cup \{u\}) \geq f(\mathcal{B}) - f(\mathcal{B} \cup \{u\})$$



"diminishing returns"

When is greedy search good enough?

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

- $1/(1+P)$-optimal when f is supermodular
  [Fisher'78, Conforti'84]

- LQR cost is NOT supermodular
  [Tzoumas'16, Olshevsky'16, Singh'17, Zhang'17]

α-supermodularity

Definition (Supermodularity)

For $\mathcal{A} \subset \mathcal{B} \subseteq \overline{\mathcal{V}}$ and $u \in \overline{\mathcal{V}} \setminus \mathcal{B}$

$$f\left(\mathcal{A} \cup \{u\}\right) - f\left(\mathcal{A}\right) \leq f\left(\mathcal{B} \cup \{u\}\right) - f\left(\mathcal{B}\right)$$

Definition (α-supermodularity)

For $\mathcal{A} \subset \mathcal{B} \subseteq \overline{\mathcal{V}}$, $u \in \overline{\mathcal{V}} \setminus \mathcal{B}$, and $\alpha \in \mathbb{R}_+$

$$f(\mathcal{A} \cup \{u\}) - f(\mathcal{A}) \leq \alpha \Big[f(\mathcal{B} \cup \{u\}) - f(\mathcal{B}) \Big]$$

- If $\alpha \geq 1$: f is supermodular
- If $\alpha < 1$: f is *approximately* supermodular

Definition (α-supermodularity)

For $\mathcal{A} \subset \mathcal{B} \subseteq \overline{\mathcal{V}}$, $u \in \overline{\mathcal{V}} \setminus \mathcal{B}$, and $\alpha \in \mathbb{R}_+$

$$f(\mathcal{A} \cup \{u\}) - f(\mathcal{A}) \leq \alpha \left[f(\mathcal{B} \cup \{u\}) - f(\mathcal{B}) \right]$$

- If $\alpha \geq 1$: f is supermodular
- If $\alpha < 1$: f is *approximately* supermodular
- If $\mathcal{C} = \{\mathcal{S} \subseteq \overline{\mathcal{V}} \mid |\mathcal{S}| \leq s\}$, then $f(\mathcal{G}) \leq (1 - e^{-\alpha}) f(\mathcal{X}^\star)$
  [see, e.g., Chamon-Ribeiro'16]

Greedy α-supermodular minimization

Theorem (Chamon-Amice-Ribeiro)

If $\mathcal{C}$ is an intersection of P matroids and f is normalized, monotone decreasing, and α-supermodular, then

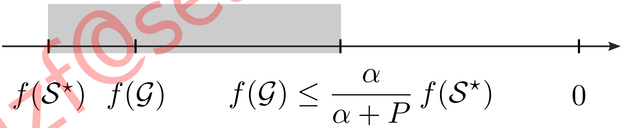
$$f(\mathcal{S}^\star) \leq f(\mathcal{G}) \leq \frac{\alpha}{\alpha + P} f(\mathcal{S}^\star) < 0$$

Greedy α-supermodular minimization

Theorem (Chamon-Amice-Ribeiro)

If $\mathcal{C}$ is an intersection of P matroids and f is normalized, monotone decreasing, and α-supermodular, then

$$f(\mathcal{S}^\star) \leq f(\mathcal{G}) \leq \frac{\alpha}{\alpha + P} f(\mathcal{S}^\star) < 0$$



What scheduling constraints are tractable?

How close can we get to the optimal schedule?

When is greedy search good enough?

How good is greedy LQR scheduling?

What is α for the LQR cost?

- Combinatorial problem

$$\bar{\alpha} = \min_{\substack{\mathcal{A} \subset \mathcal{B} \subseteq \overline{\mathcal{V}} \\ u \in \overline{\mathcal{V}} \setminus \mathcal{B}}} \frac{f\left(\mathcal{A} \cup \{u\}\right) - f\left(\mathcal{A}\right)}{f\left(\mathcal{B} \cup \{u\}\right) - f\left(\mathcal{B}\right)}$$

What is α for the LQR cost?

Proposition (Chamon-Amice-Ribeiro)

Let $\boldsymbol{A}$ be full rank. The LQR cost is α-supermodular with

$$\alpha \geq \frac{\lambda_{\min}\left[\tilde{\boldsymbol{P}}_1^{-1}(\emptyset)\right]}{\lambda_{\max}\left[\tilde{\boldsymbol{P}}_1^{-1}(\overline{\mathcal{V}}) + \sum_{i \in \mathcal{V}_0} r_i^{-1} \tilde{\boldsymbol{b}}_i \tilde{\boldsymbol{b}}_i^T\right]} > 0$$

for $\tilde{\boldsymbol{P}}_1(\mathcal{S}) = \boldsymbol{H}\boldsymbol{P}_1(\mathcal{S})\boldsymbol{H}$, $\tilde{\boldsymbol{b}}_{i,k} = \boldsymbol{H}^{-1}\boldsymbol{b}_{i,0}$, *and* $\boldsymbol{H} = \left(\boldsymbol{A}\boldsymbol{\Sigma}_0\boldsymbol{A}^T\right)^{1/2}$.

When is α large for the LQR cost?

$$\sum_{i \in \mathcal{V}_0} r_{i,0}^{-1} \tilde{\boldsymbol{b}}_i \tilde{\boldsymbol{b}}_i^T \text{ is small} \quad \text{and} \quad \lambda_{\text{max}} \left[\tilde{\boldsymbol{P}}_1^{-1} \left(\overline{\mathcal{V}} \right) \right] \approx \lambda_{\text{min}} \left[\tilde{\boldsymbol{P}}_1^{-1} (\emptyset) \right]$$

When is α large for the LQR cost?

$$\sum_{i \in \mathcal{V}_0} r_{i,0}^{-1} \tilde{\boldsymbol{b}}_i \tilde{\boldsymbol{b}}_i^T \text{ is small} \quad \text{and} \quad \lambda_{\max}\left[\tilde{\boldsymbol{P}}_1^{-1}\left(\overline{\mathcal{V}}\right)\right] \approx \lambda_{\min}\left[\tilde{\boldsymbol{P}}_1^{-1}(\emptyset)\right]$$

- $\boldsymbol{R} \gg \boldsymbol{Q}$: large α, better guarantees

When is α large for the LQR cost?

$$\sum_{i \in \mathcal{V}_0} r_{i,0}^{-1} \tilde{\boldsymbol{b}}_i \tilde{\boldsymbol{b}}_i^T \text{ is small} \quad \text{and} \quad \lambda_{\text{max}}\left[\tilde{\boldsymbol{P}}_1^{-1}\left(\overline{\mathcal{V}}\right)\right] \approx \lambda_{\text{min}}\left[\tilde{\boldsymbol{P}}_1^{-1}(\emptyset)\right]$$

- $\boldsymbol{R} \ggg \boldsymbol{Q}$: large α, better guarantees
- $\boldsymbol{R} \lll \boldsymbol{Q}$, the LQR cost really only distinguishes controllable from uncontrollable sets ($\boldsymbol{R} = \boldsymbol{0} \Rightarrow$ *dead-beat controller*)

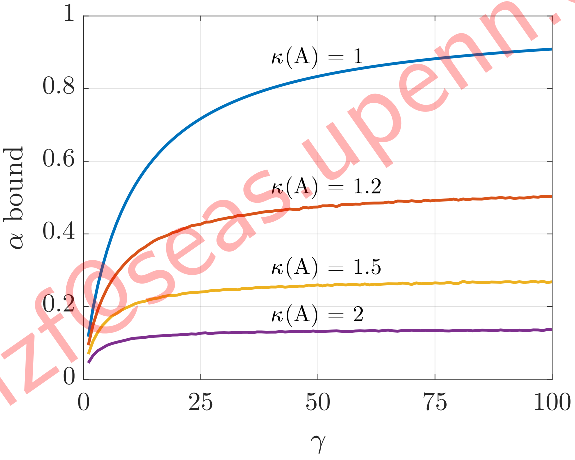
When is α large for the LQR cost?

$$\sum_{i\in\mathcal{V}_0} r_{i,0}^{-1}\tilde{\boldsymbol{b}}_i\tilde{\boldsymbol{b}}_i^T \text{ is small} \quad \text{and} \quad \lambda_{\text{max}}\left[\tilde{\boldsymbol{P}}_1^{-1}\left(\overline{\mathcal{V}}\right)\right] \approx \lambda_{\text{min}}\left[\tilde{\boldsymbol{P}}_1^{-1}(\emptyset)\right]$$

- $\boldsymbol{R} \ggcurly \boldsymbol{Q}$: large α, better guarantees
- $\boldsymbol{R} \llcurly \boldsymbol{Q}$, the LQR cost really only distinguishes controllable from uncontrollable sets ($\boldsymbol{R} = \boldsymbol{0} \Rightarrow$ *dead-beat controller*)
- Also depends on the condition numbers of $\boldsymbol{A}$ and $\boldsymbol{\Sigma}_0$

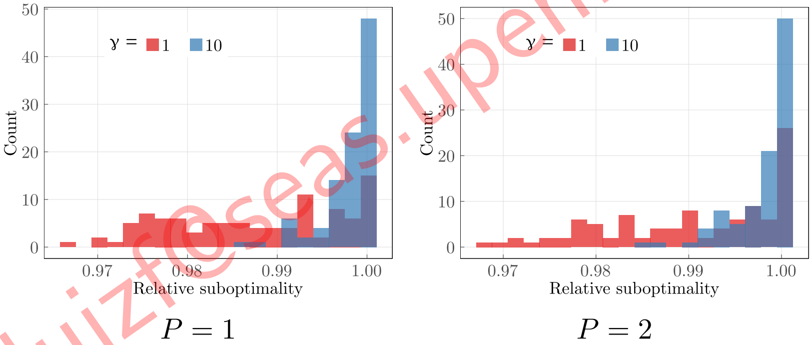
What is α for the LQR cost?

- $|\mathcal{V}| = 100$, $\boldsymbol{B} = \boldsymbol{I}$, and $\boldsymbol{R} = \gamma \boldsymbol{Q}$



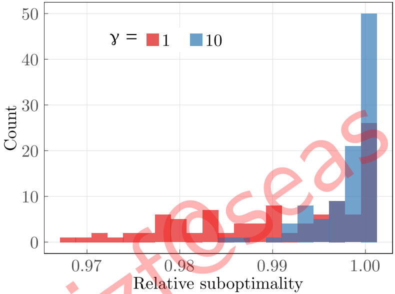
Guarantees for greedy LQR scheduling

- $|\mathcal{V}| = 7$, $\boldsymbol{B} = \boldsymbol{I}$, $N = 4$ iterations, and $\boldsymbol{R} = \gamma \boldsymbol{Q}$

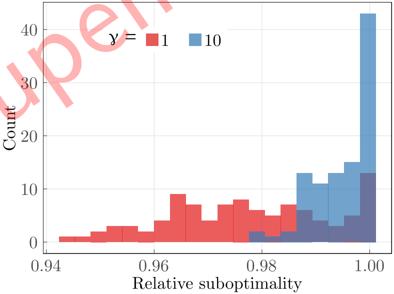


Guarantees for greedy LQR scheduling

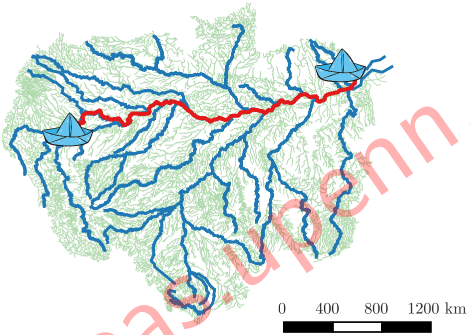
- $|\mathcal{V}| = 7$, $\boldsymbol{B} = \boldsymbol{I}$, $N = 4$ iterations, and $\boldsymbol{R} = \gamma \boldsymbol{Q}$



$P = 2$

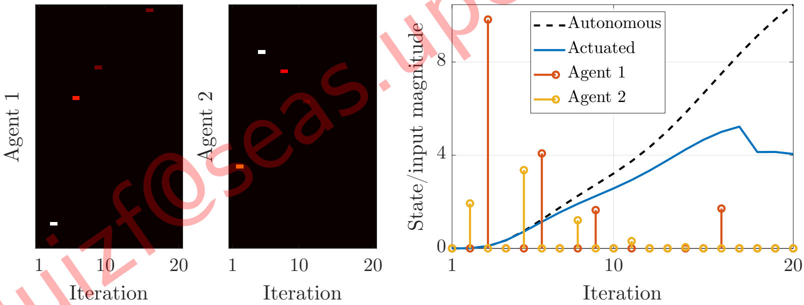


$P = 3$



- Dispatch two agents on the Amazon river to control spill
  - Predetermined route
  - Limited # of actions
  - Duty cycle

- Schedule constraints
  - ≤ 5 actuations per agent
  - Cool-off period ≥ 2 iterations



The control scheduling problem

Conclusion

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?
   Matroid intersections

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?
   Matroid intersections
2. How close can we get to the optimal schedule?

Conclusion

The control scheduling problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \overline{\mathcal{V}}}{\text{minimize}} \quad & f(\mathcal{S}) \\ \text{subject to} \quad & \mathcal{S} \in \mathcal{C} \end{aligned}$$

1. What scheduling constraints are tractable?
   Matroid intersections
2. How close can we get to the optimal schedule?
   Greedy LQR scheduling is $\dfrac{\alpha}{\alpha + P}$-optimal

Control Scheduling Subject to Matroid Constraints

Luiz F. O. Chamon,
Alexandre Amice, and Alejandro Ribeiro

More details: http://www.seas.upenn.edu/∼luizf