Near-Optimality of Greedy Set Selection in the Sampling of Graph Signals

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Greedy Sampling of Graph Signals

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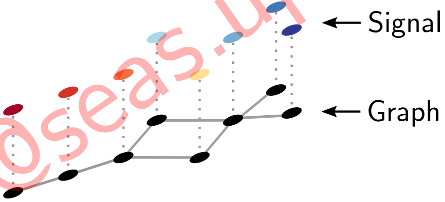
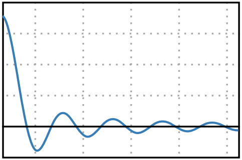
Definition

It's a signal that comes with a graph.



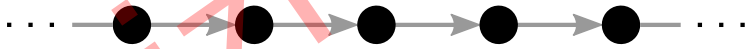
Old dog

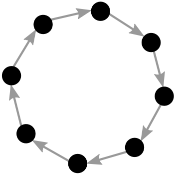
In traditional signal processing, we only look at the values . . .



$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

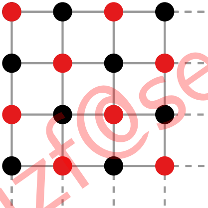
. . . because the structure is implicit

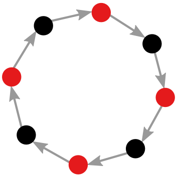




Greedy Sampling of Graph Signals

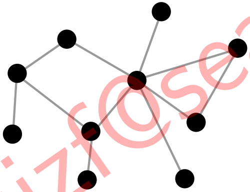
- **Classical signals**: sampling is "easy" on regular domains

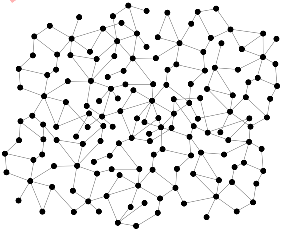




So what? I already know how to do this...

- **Classical signals**: sampling is "easy" on regular domains
- **Graph signals**: not so easy (combinatorial)





Greedy Sampling of Graph Signals

Definition

Pick nodes one at a time by always choosing the one that most improves interpolation at each step.

function $\textsc{GreedySampling}(\ell)$

$\mathcal{G}_0 = \{\}$

for $j = 1, \ldots, \ell$

$u = \operatorname{argmin}_{s \in \mathcal{V} \setminus \mathcal{G}_{j-1}} \mathsf{MSE}\left(\mathcal{G}_{j-1} \cup \{s\}\right)$

$\mathcal{G}_j = \mathcal{G}_{j-1} \cup \{u\}$

end

end

- Pros:
  - Low complexity
  - Sequential
  - Empirically successful
- Cons:
  - Is it guaranteed to be close to optimal?

Greedy sampling

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  - Low complexity
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- Cons:
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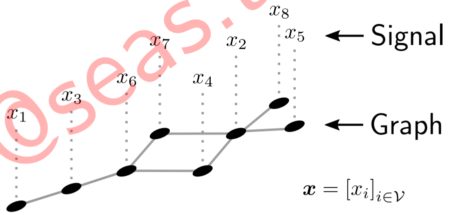
Greedy sampling is guaranteed to do a good job minimizing the interpolation MSE

Graph signal formalism

- A graph signal is a pair $(\mathbb{G}, \boldsymbol{x})$
  - A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$
    - $\boldsymbol{A}$ is a matrix representation of $\mathbb{G}$ (e.g., adjacency, Laplacian)
    - **Assumption (Parseval):** $\boldsymbol{A}$ is normal, i.e., $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T$
  - A signal $\boldsymbol{x} \in \mathbb{R}^n$ defined over $\mathcal{V}$



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- Graph Fourier Transform

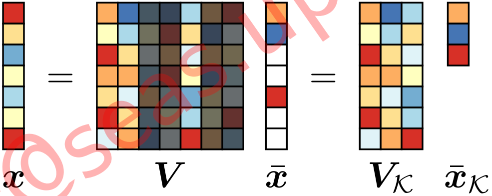
$$\bar{\boldsymbol{x}} = \boldsymbol{V}^T \boldsymbol{x} \quad \longleftrightarrow \quad \boldsymbol{x} = \boldsymbol{V} \bar{\boldsymbol{x}}$$

- A signal is $\mathcal{K}$-bandlimited if $\bar{x}$ is $\mathcal{K}$-sparse: $\bar{\boldsymbol{x}}_{\mathcal{V} \setminus \mathcal{K}} = \boldsymbol{0}$

$$\boldsymbol{x} = \boldsymbol{V}_{\mathcal{K}} \bar{\boldsymbol{x}}_{\mathcal{K}}$$

$\mathcal{K}$-bandlimited graph signal

- $\mathcal{K} = \{1, 2, 5\}$



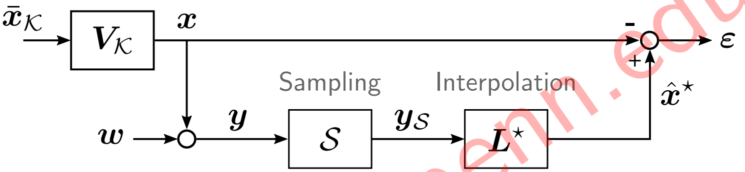
Stochastic graph signal

- **Signal**: $\bar{\boldsymbol{x}}_{\mathcal{K}}$ is a zero-mean RV with covariance $\boldsymbol{\Lambda} = \sigma_x^2 \boldsymbol{I}$

$$\boldsymbol{x} = \boldsymbol{V}_{\mathcal{K}} \bar{\boldsymbol{x}}_{\mathcal{K}}$$

- **Noise**: $\boldsymbol{w}$ is a zero-mean RV with covariance $\boldsymbol{\Lambda}_w = \sigma_w^2 \boldsymbol{I}$

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{w}$$



- Optimal interpolator:

$$L^\star C \left(V_\mathcal{K} \Lambda V_\mathcal{K}^T + \Lambda_w \right) C^T = V_\mathcal{K} \Lambda V_\mathcal{K}^T C^T$$

- Optimal interpolation MSE:

$$\text{MSE}(\mathcal{S}) = \mathbb{E} \left\| x - \hat{x}^\star \right\|^2 = \text{Tr} \left[\left(\sigma_x^{-2} I + \sigma_w^{-2} \sum_{i \in \mathcal{S}} v_i v_i^T \right)^{-1} \right]$$

The sampling set selection problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{minimize}} \quad & \text{MSE}(\mathcal{S}) \\ \text{subject to} \quad & |\mathcal{S}| = k \end{aligned}$$

- Set function minimization with cardinality constraint

Theorem ([NWF, 1978])

Let $\mathcal{S}^\star$ be the optimal solution of the problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{minimize}} \quad & f\left(\mathcal{S}\right) \\ \text{subject to} \quad & |\mathcal{S}| = k \end{aligned}$$

and $\mathcal{G}$ be its greedy solution. If f is (i) monotone decreasing and (ii) supermodular, then

$$\frac{f(\mathcal{G}) - f(\mathcal{S}^\star)}{f(\{\}) - f(\mathcal{S}^\star)} \leq e^{-1} \approx 0.37.$$

Theorem ([NWF, 1978])

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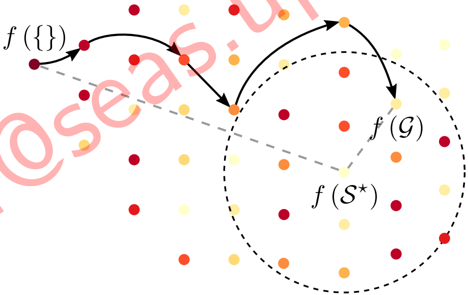
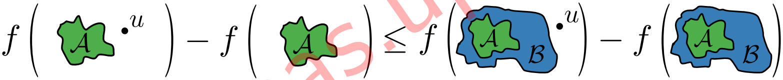


Supermodularity

Definition (Supermodularity)

For $\mathcal{A} \subseteq \mathcal{B}$ and $u \notin \mathcal{B}$,

$$f\left(\mathcal{A} \cup \{u\}\right) - f\left(\mathcal{A}\right) \leq f\left(\mathcal{B} \cup \{u\}\right) - f\left(\mathcal{B}\right)$$



"diminishing returns"

Which f are supermodular?

- $\mathsf{MSE}(\mathcal{S})$ is NOT supermodular
- $\log\det$ of the error covariance matrix is supermodular

Now what?

Definition (Supermodularity)

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$$f\left(\mathcal{A} \cup \{u\}\right) - f\left(\mathcal{A}\right) \leq f\left(\mathcal{B} \cup \{u\}\right) - f\left(\mathcal{B}\right)$$

Now what?

Definition (Approximate supermodularity or α-supermodularity)

For $\mathcal{A} \subseteq \mathcal{B}$ and $u \notin \mathcal{B}$, and $\alpha \in [0, 1]$

$$f(\mathcal{A} \cup \{u\}) - f(\mathcal{A}) \leq \alpha \Big[f(\mathcal{B} \cup \{u\}) - f(\mathcal{B}) \Big]$$

- If $\alpha = 1$, then f is supermodular

Theorem

Let $\mathcal{S}^\star$ be the optimal solution of the problem

$$\begin{aligned} \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{minimize}} \quad & f\left(\mathcal{S}\right) \\ \text{subject to} \quad & \left|\mathcal{S}\right| = k \end{aligned}$$

and $\mathcal{G}_\ell$ be the ℓ-th iteration of a greedy solution. If f is (i) monotone decreasing and (ii) α-supermodular, then

$$\frac{f(\mathcal{G}_\ell) - f(\mathcal{S}^\star)}{f(\{\}) - f(\mathcal{S}^\star)} \leq e^{-\alpha \ell / k}.$$

Greedy α-supermodular minimization

Theorem

If f is (i) monotone decreasing and (ii) α-supermodular, then

$$\frac{f(\mathcal{G}_\ell) - f(\mathcal{S}^\star)}{f(\{\}) - f(\mathcal{S}^\star)} \leq e^{-\alpha\ell/k}.$$

- For $\ell = k$ and $\alpha = 1$, we recover the classical greedy result
- If $\alpha < 1$, then e^{-1} is recovered for $\ell = \alpha^{-1}k$
- Evaluating α is NP-hard

What is α for the MSE?

Theorem

The MSE($\mathcal{S}$) *is* α*-supermodular with*

$$\alpha \geq \frac{1+2\gamma}{(1+\gamma)^4}, \quad \text{for} \quad \gamma = \frac{\sigma_x^2}{\sigma_w^2}.$$

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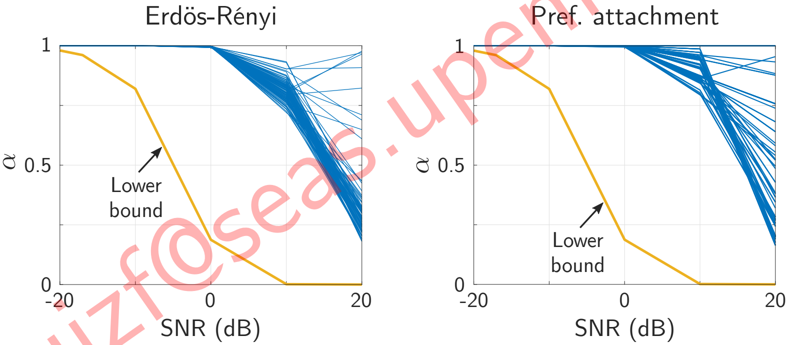
$$\alpha \geq \frac{1+2\gamma}{(1+\gamma)^4}, \quad \text{for} \quad \gamma = \frac{\sigma_x^2}{\sigma_w^2}.$$

- $\alpha \to 1$ as $\gamma \to 0$
- $\alpha \to 0$ as $\gamma \to \infty$
  - In the noiseless case, almost every $\mathcal{S}$ with $|\mathcal{S}| \geq |\mathcal{K}|$ yields perfect reconstruction

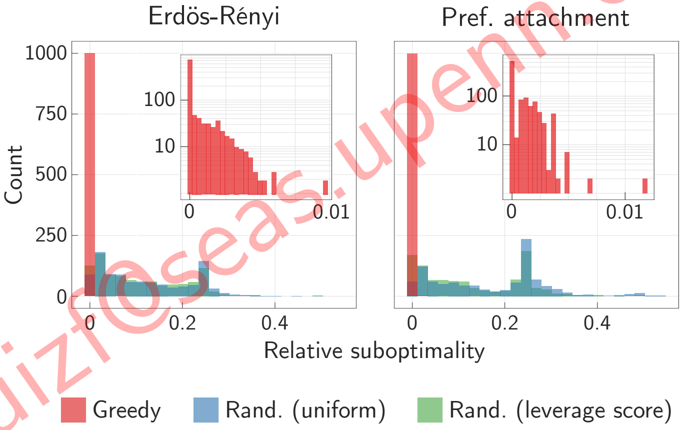
Simulations: α for the MSE

- $n = 10$ nodes, $|\mathcal{K}| = 4$, and 100 realizations

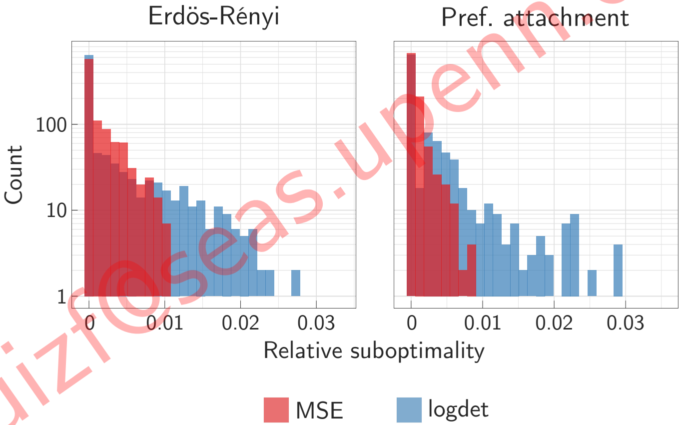


Simulations: greedy sampling

- MSE (10 nodes, $|\mathcal{S}| = |\mathcal{K}| = 4$, and SNR $= 20$ dB)

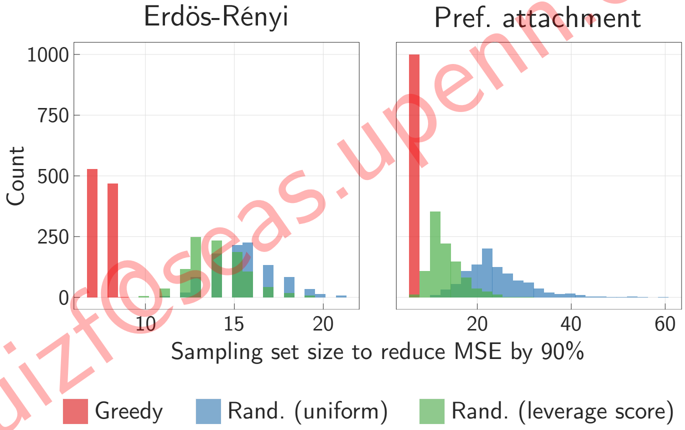


- MSE vs $\log\det$ (10 nodes, $|\mathcal{S}| = |\mathcal{K}| = 4$, and SNR $= 20$ dB)



Simulations: greedy sampling

- 100 nodes, $|\mathcal{K}| = 7$, and SNR $= 20$ dB



Conclusion

- Graph signal sampling is useful, but it's hard
- Interpolation MSE is not supermodular, but almost
- Greedy sampling set selection is efficient and has a guaranteed near-optimal performance

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More details: `http://www.seas.upenn.edu/~luizf`