Finite-Precision Effects on Graph Filters

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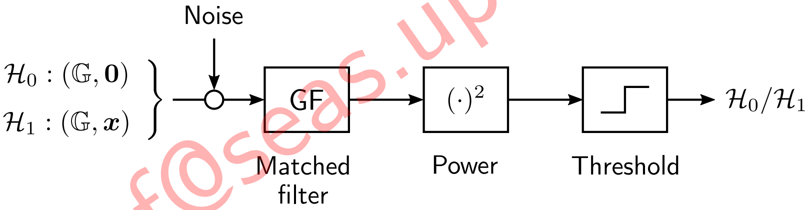
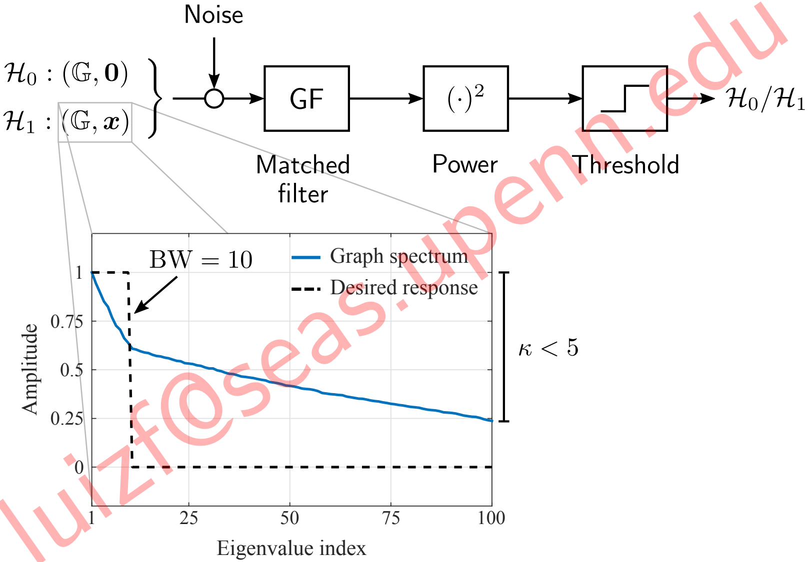
Why?

- Graph filters are important tools in GSP [Narang'12, Sandryhaila'13, Shuman'13, Segarra'17, Isufi'17, Teke'17, Defferrard'17]
- Filtering is done by finite precision machines (CPUs, GPUs, FPGAs. . .)
- This can cause serious problems in GSP

- Graph signal detection using matched filter (LDA)



LDA using graph filters



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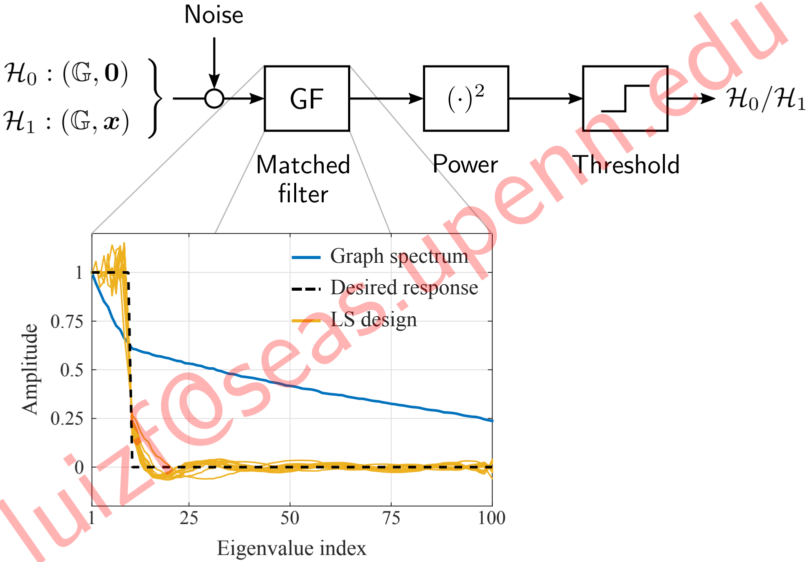
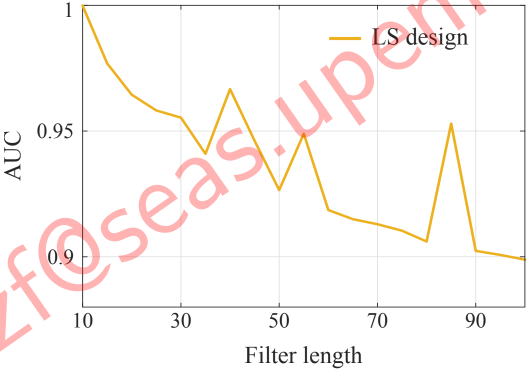


- 32 bits floating-point (single precision)



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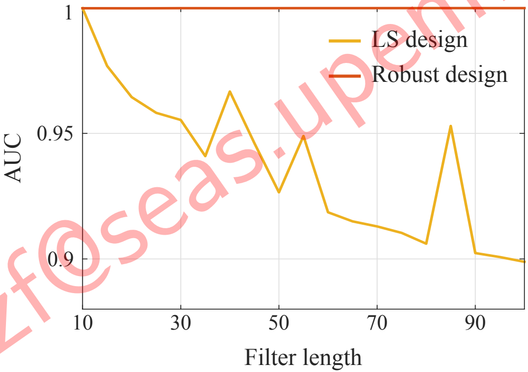

Definitions

Graph filters

Quantization noise

Graph filters in fixed-point arithmetic

Robust graph filter design

Graph filters

- A graph signal is a pair $(\boldsymbol{S}, \boldsymbol{x})$
  - $\boldsymbol{x} \in \mathbb{R}^n$ is the signal
  - $\boldsymbol{S} \in \mathbb{R}^{n \times n}$ is the shift operator
    - **Assumption:** $\boldsymbol{S} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^{-1}$ and $\lambda_i \in \mathbb{R}$

- Linear shift-invariant filters [Sandryhaila'13]

$$\boldsymbol{y} = \left(\sum_{k=0}^{L-1} h_k \boldsymbol{S}^k \right) \boldsymbol{x}$$

Graph filter design

- LS design for a desired response $\boldsymbol{d}$: [Sandryhaila'13, Shuman'13, Segarra'17]

$$\boldsymbol{h}^\star \in \underset{\boldsymbol{h}}{\operatorname{argmin}} \|\boldsymbol{d} - \boldsymbol{\Psi}\boldsymbol{h}\|_2^2$$

$$\boldsymbol{h} = [h_0 \ \cdots \ h_{L-1}]^T \quad \text{and} \quad [\boldsymbol{\Psi}]_{ij} = \lambda_i^{j-1}$$

- Vandermonde system of equation: extremely ill-conditioned

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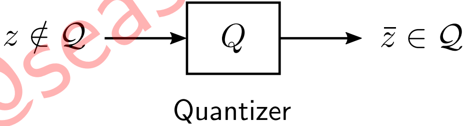
Quantization error

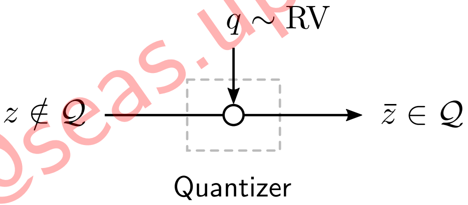
- Finite-precision machines can only represent a finite set of numbers $\mathcal{Q}$
- If $z \notin \mathcal{Q}$, the machine replaces it by $\bar{z}$ (quantization)



Quantization error

- Finite-precision machines can only represent a finite set of numbers $\mathcal{Q}$

- If $z \notin \mathcal{Q}$, the machine replaces it by $\bar{z} = z + q$ (quantization)



- Fixed-point $QB.K$: $q \sim$ Uniform$([-2^{-K-1}, 2^{-K-1}])$

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- Quantized graph filtering

$$\boldsymbol{y} = \left(\sum_{k=0}^{L-1} h_k \boldsymbol{S}^k \right) \boldsymbol{x}$$

Graph filters in fixed-point arithmetic

- Quantized graph filtering

$$\boldsymbol{y} = \left(\sum_{k=0}^{L-1} h_k \boldsymbol{S}^k\right) \boldsymbol{x} \Rightarrow$$

$$\bar{\boldsymbol{y}} = \underbrace{Q\left[Q\left[Q[h_0 \boldsymbol{x}] + Q\left[h_1 Q[\boldsymbol{S}\boldsymbol{x}]\right]\right] + \dots\right]}_{\text{Quantize after each MAC}}$$

Graph filters in fixed-point arithmetic

- Quantized graph filtering

$$\boldsymbol{y} = \left(\sum_{k=0}^{L-1} h_k \boldsymbol{S}^k\right) \boldsymbol{x} \Rightarrow$$

$$\bar{\boldsymbol{y}} = Q\left[Q\left[Q[h_0\boldsymbol{x}] + Q\left[h_1 Q[\boldsymbol{S}\boldsymbol{x}]\right]\right] + \dots\right]$$

$$= (h_0\boldsymbol{x} + \boldsymbol{w}_0) + \left[h_1(\boldsymbol{S}\boldsymbol{x} + \boldsymbol{v}_1) + \boldsymbol{w}_1\right] + \dots + \boldsymbol{w}_{L-1}$$

Graph filters in fixed-point arithmetic

- Quantized graph filtering

$$\bar{\boldsymbol{x}}_{k+1} = \boldsymbol{S}\bar{\boldsymbol{x}}_k + \boldsymbol{v}_k \qquad \text{(Shift)}$$

$$\bar{\boldsymbol{y}} = \sum_{k=0}^{L-1} h_k \bar{\boldsymbol{x}}_k + \boldsymbol{w} \qquad \text{(Filtering)}$$

- $\{\boldsymbol{v}_k, \boldsymbol{w}\}$ represent the overall quantization noises

Proposition

In $QB.K$ arithmetic, the output MSE due to quantization is

$$\mathbb{E}\,\|\bar{\boldsymbol{y}} - \boldsymbol{y}\|_2^2 = \left(L + \|\boldsymbol{P}\boldsymbol{H}\|_F^2\right) n\sigma^2,$$

where $\sigma^2 = 2^{-2K}/12$, $\boldsymbol{P}$ is a submatrix of $\boldsymbol{\Psi}$, and $\boldsymbol{H}$ is the Hankel matrix of the filter coefficients.

Graph filters in fixed-point arithmetic

$$\mathbb{E}\,\|\bar{\boldsymbol{y}} - \boldsymbol{y}\|_2^2 = \left(L + \|\boldsymbol{P}\boldsymbol{H}\|_F^2\right) n \cdot \frac{2^{-2K}}{12}$$

- When are graph filters susceptible to numerical issues?

Graph filters in fixed-point arithmetic

$$\mathbb{E}\,\|\bar{\boldsymbol{y}} - \boldsymbol{y}\|_2^2 = \left(L + \|\boldsymbol{PH}\|_F^2 \right) n \cdot \frac{2^{-2K}}{12}$$

- When are graph filters susceptible to numerical issues?
  - Short transition band

Graph filters in fixed-point arithmetic

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  - Large spectral gain

Graph filters in fixed-point arithmetic

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- When are graph filters susceptible to numerical issues?
  - Short transition band
  - Large spectral gain
  - Large shift operator spectral radius

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Goal

Given a shift operator $\boldsymbol{S}$, design a graph filter with response $\boldsymbol{d}$ robust to round-off error.

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Proposition

In $QB.K$ arithmetic, the output MSE due to quantization is

$$\mathbb{E}\left\|\bar{\boldsymbol{y}}-\boldsymbol{y}\right\|_2^2=\left(L+\left\|\boldsymbol{P}\boldsymbol{H}\right\|_F^2\right)n\sigma^2,$$

where $\sigma^2=2^{-2K}/12$, $\boldsymbol{P}$ is a submatrix of $\boldsymbol{\Psi}$, and $\boldsymbol{H}$ is the Hankel matrix of the filter coefficients.

Robust graph filter design

$$\underset{\boldsymbol{h} \in \mathbb{R}^L}{\text{minimize}} \quad \|\boldsymbol{d} - \boldsymbol{\Psi} \boldsymbol{h}\|_2^2 + \eta^2 \left(h_0^2 + \|\boldsymbol{P}\boldsymbol{H}\|_F^2 \right)$$

- Shortest length with small error: design for different lengths
- Stable solver: $\boldsymbol{\Psi}$ is ill-conditioned

Bit-accurate experiments

- LS design

$$\underset{\boldsymbol{h} \in \mathbb{R}^L}{\text{minimize}} \quad \|\boldsymbol{d} - \boldsymbol{\Psi h}\|_2^2$$

- Robust design

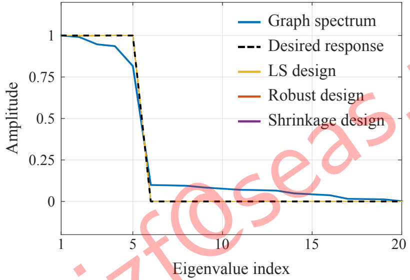
$$\underset{\boldsymbol{h} \in \mathbb{R}^L}{\text{minimize}} \quad \|\boldsymbol{d} - \boldsymbol{\Psi h}\|_2^2 + \eta_r^2 \left(h_0^2 + \|\boldsymbol{PH}\|_F^2 \right)$$

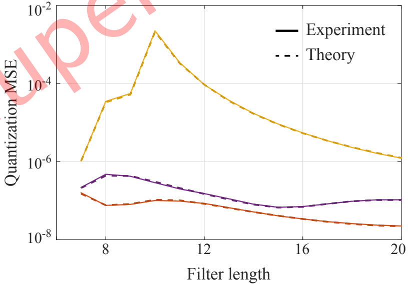
- Shrinkage design

$$\underset{\boldsymbol{h} \in \mathbb{R}^L}{\text{minimize}} \quad \|\boldsymbol{d} - \boldsymbol{\Psi h}\|_2^2 + \eta_s^2 \|\boldsymbol{h}\|_2^2$$

Bit-accurate experiments

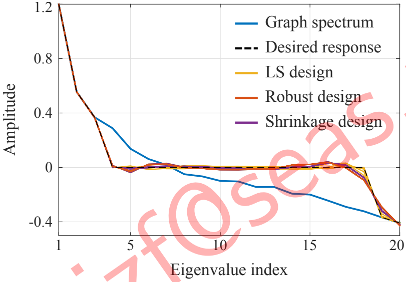
- Simple scenario in signed $Q13.18$

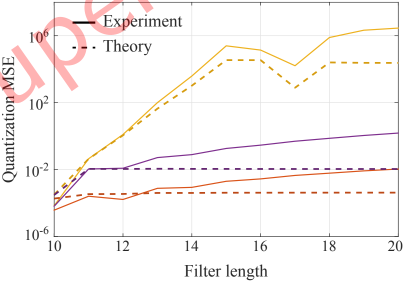




Bit-accurate experiments

- $\rho(\boldsymbol{S}) = 1.2$ in signed $Q43.20$





Conclusion

- Finite precision can have catastrophic effects on signal processing systems
- Graph filters are particularly susceptible to finite precision
- Robust design of graph filters requires proper regularization and stable solvers

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More details: `http://www.seas.upenn.edu/~luizf`