Universal Bounds for the Sampling of Graph Signals

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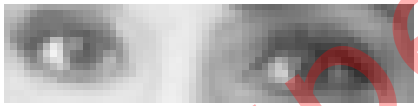
Face recognition

Jane Doe

Face recognition

Face recognition

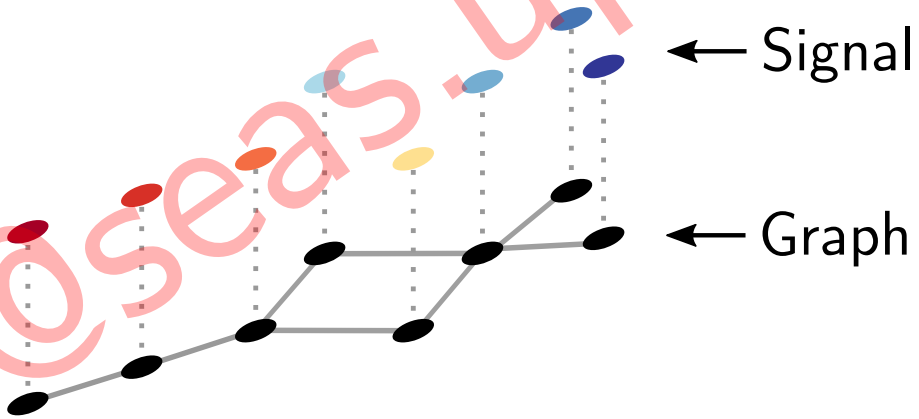


Face recognition

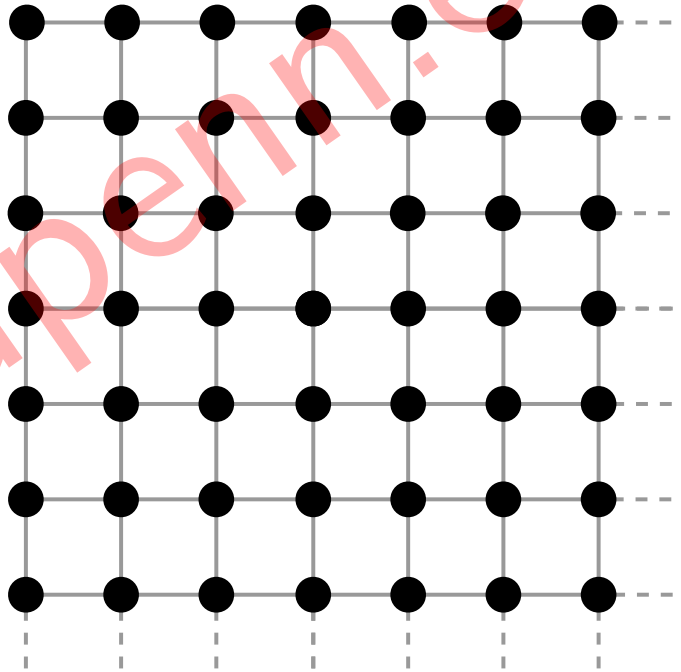
What is a graph signal?

Definition

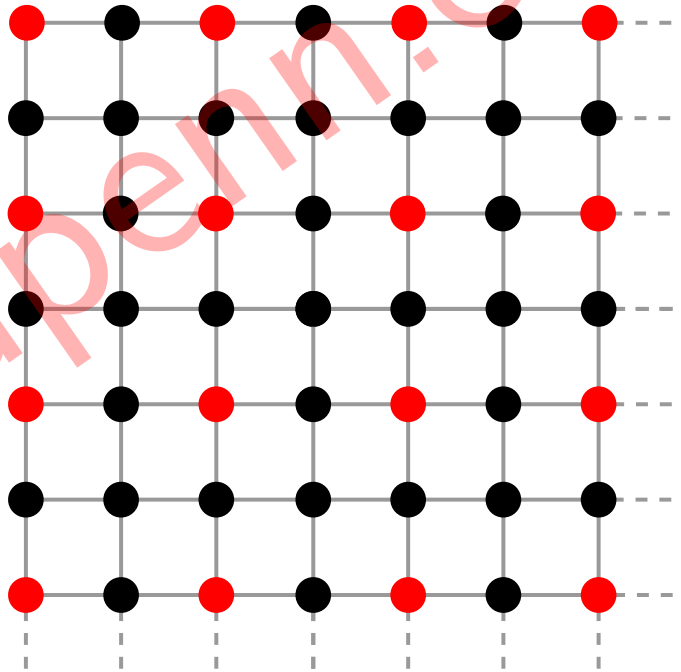
- It's a signal that comes with a graph
- The graph represents *proximity* between the signal samples



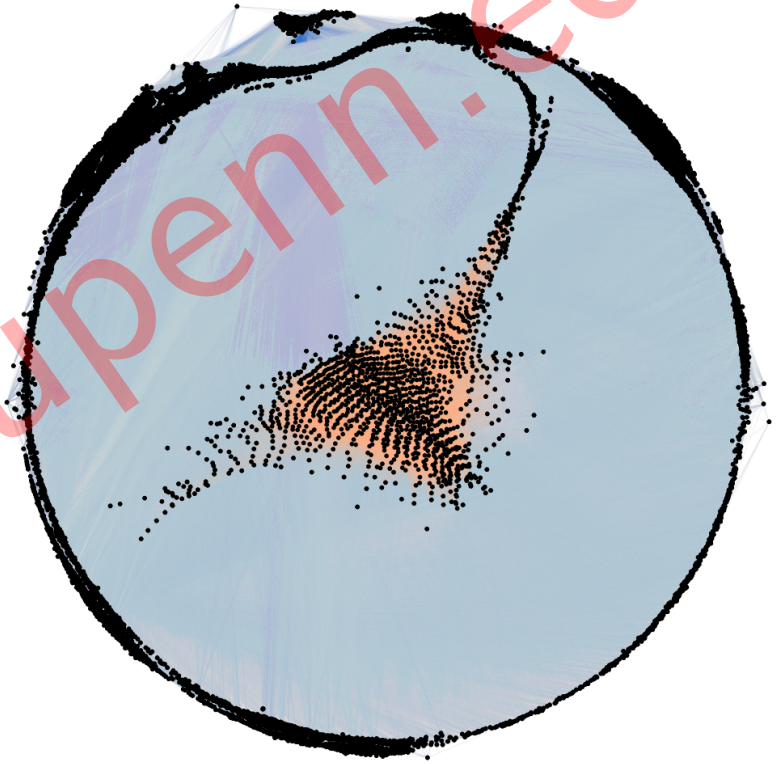
Face recognition: which graph?



Face recognition: which graph?



Face recognition: which graph?



Why universal bounds?

- Understand which factors influence the reconstruction performance
- Gauge the interpolation quality a sampling set or heuristic
- Performance benchmark

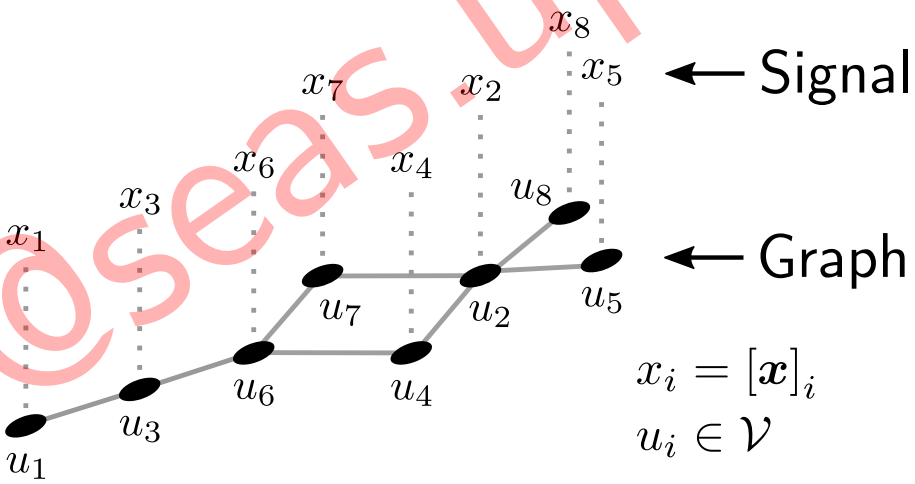
Universal sampling bounds

Greedy sampling evaluation

Sampled kernel PCA

Graph signal formalism

- A graph signal is a pair $(\mathbb{G}, \boldsymbol{x})$
  - A graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$
    - $\boldsymbol{A}$ is a matrix representation of $\mathbb{G}$ (e.g., adjacency, Laplacian)
    - **Assumption (Parseval):** $\boldsymbol{A}$ is normal, i.e., $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}^T$
  - A signal $\boldsymbol{x} \in \mathbb{R}^n$ defined over $\mathcal{V}$



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- Graph Fourier Transform

$$\bar{\boldsymbol{x}} = \boldsymbol{V}^T \boldsymbol{x} \quad \longleftrightarrow \quad \boldsymbol{x} = \boldsymbol{V}\bar{\boldsymbol{x}}$$

- A signal is $\mathcal{K}$-bandlimited if $\bar{\boldsymbol{x}}$ is $\mathcal{K}$-sparse: $\bar{\boldsymbol{x}}_{\mathcal{V}\setminus\mathcal{K}} = \boldsymbol{0}$

$$\boldsymbol{x} = \boldsymbol{V}_{\mathcal{K}}\bar{\boldsymbol{x}}_{\mathcal{K}}$$

Stochastic graph signal

- **Signal**: $\bar{\boldsymbol{x}}_{\mathcal{K}}$ is a zero-mean RV with covariance $\boldsymbol{\Lambda} = \sigma_x^2 \boldsymbol{I}$

$$\boldsymbol{x} = \boldsymbol{V}_{\mathcal{K}} \bar{\boldsymbol{x}}_{\mathcal{K}}$$

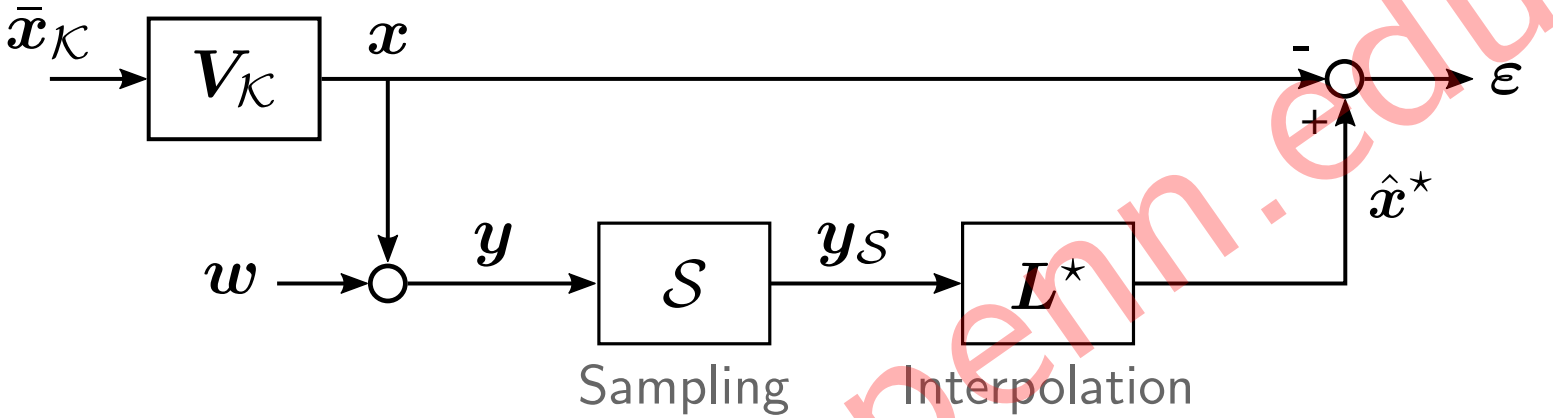
- **Noise**: $\boldsymbol{w}$ is a zero-mean RV with covariance $\boldsymbol{\Lambda}_w = \sigma_w^2 \boldsymbol{I}$

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{w}$$

- **SNR**:

$$\gamma = \frac{\sigma_x^2}{\sigma_w^2}$$

Sampling and interpolation



- Optimal interpolator:

$$\boldsymbol{L}^\star \boldsymbol{C} \left(\boldsymbol{V}_{\mathcal{K}} \boldsymbol{\Lambda} \boldsymbol{V}_{\mathcal{K}}^T + \boldsymbol{\Lambda}_w \right) \boldsymbol{C}^T = \boldsymbol{V}_{\mathcal{K}} \boldsymbol{\Lambda} \boldsymbol{V}_{\mathcal{K}}^T \boldsymbol{C}^T$$

- Optimal interpolation MSE:

$$\text{MSE}(\mathcal{S}) = \mathbb{E} \left\| \boldsymbol{x} - \boldsymbol{x}^\star \right\|^2 = \sigma_x^2 \, \text{Tr} \left[\left(\boldsymbol{I} + \gamma \sum_{i \in \mathcal{S}} \boldsymbol{v}_i \boldsymbol{v}_i^T \right)^{-1} \right]$$

Universal performance bound

Theorem

For any sampling set $\mathcal{S}$, it holds that

$$\text{MSE}(\mathcal{S}) \geq \frac{|\mathcal{K}|^2 \sigma_x^2}{|\mathcal{K}| + \bar{\ell}_{|\mathcal{S}|}}$$

where $\bar{\ell}_m$ is the sum of the m largest structural SNRs:

$$\bar{\ell}_m = \max_{\mathcal{X}:|\mathcal{X}|=m} \sum_{j \in \mathcal{X}} \gamma \left\| \boldsymbol{v}_j \right\|_2^2,$$

with $\boldsymbol{v}_j^T$ the j-th row of $\boldsymbol{V}_{\mathcal{K}}$.

Universal performance bound

Theorem

For any sampling set $\mathcal{S}$, it holds that

$$\text{MSE}(\mathcal{S}) \geq \frac{|\mathcal{K}|^2 \sigma_x^2}{|\mathcal{K}| + \bar{\ell}_{|\mathcal{S}|}}$$

- Depends on graph signal statistics and structural properties
- Universal: holds for ANY sampling set
- Increases with the signal *bandwidth*
- Decreases with $|\mathcal{S}|$ with rate dependent on the *structural SNR*

$$\ell_j = \gamma \left\| \boldsymbol{v}_j \right\|_2^2$$

Universal performance bound

Corollary

Any sampling set $\mathcal{S}$ for which $\mathrm{MSE}(\mathcal{S}) = \eta$ *satisfies*

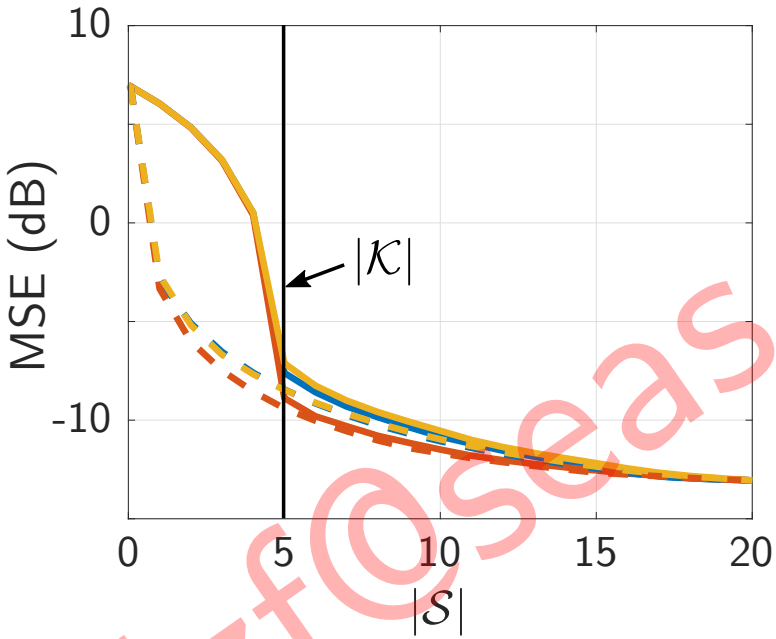
$$\bar{\ell}_{|\mathcal{S}|} \geq \frac{|\mathcal{K}|^2 \sigma_x^2 - \eta |\mathcal{K}|}{\eta}.$$

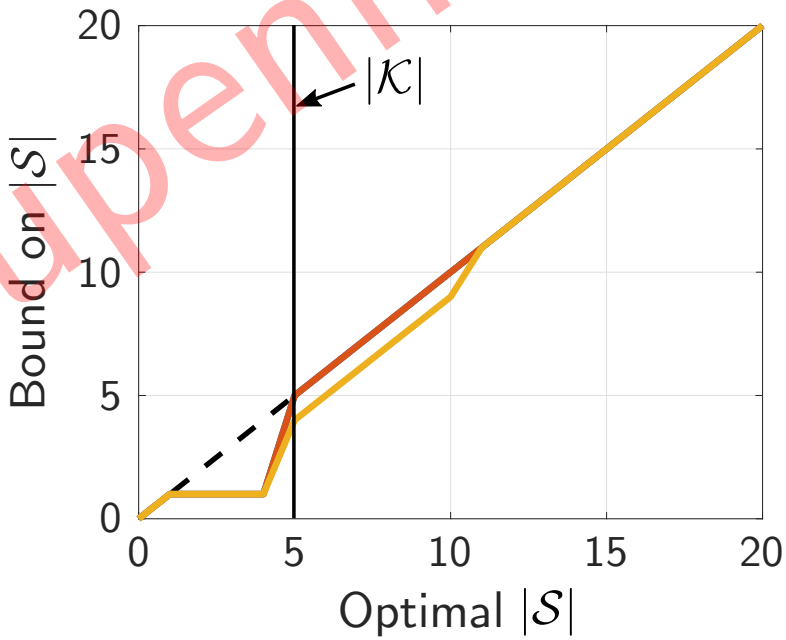
Since $\bar{\ell}_{|\mathcal{S}|} \leq |\mathcal{S}| \ell_{max}$, it also holds that

$$|\mathcal{S}| \geq \frac{|\mathcal{K}|^2 \sigma_x^2 - \eta |\mathcal{K}|}{\eta\, \ell_{max}}.$$

Universal performance bound

- $n = 20$ nodes, $|\mathcal{K}| = 5$, and SNR $= 20$ dB





Erdös-Rényi Pref. attachment Random

Greedy sampling

function GREEDYSAMPLING(k)

$\mathcal{G}_0 = \{\}$

for $j = 1, \ldots, k$

$u = \text{argmin}_{s \in \mathcal{V} \setminus \mathcal{G}_{j-1}} \text{MSE}\left(\mathcal{G}_{j-1} \cup \{s\}\right)$

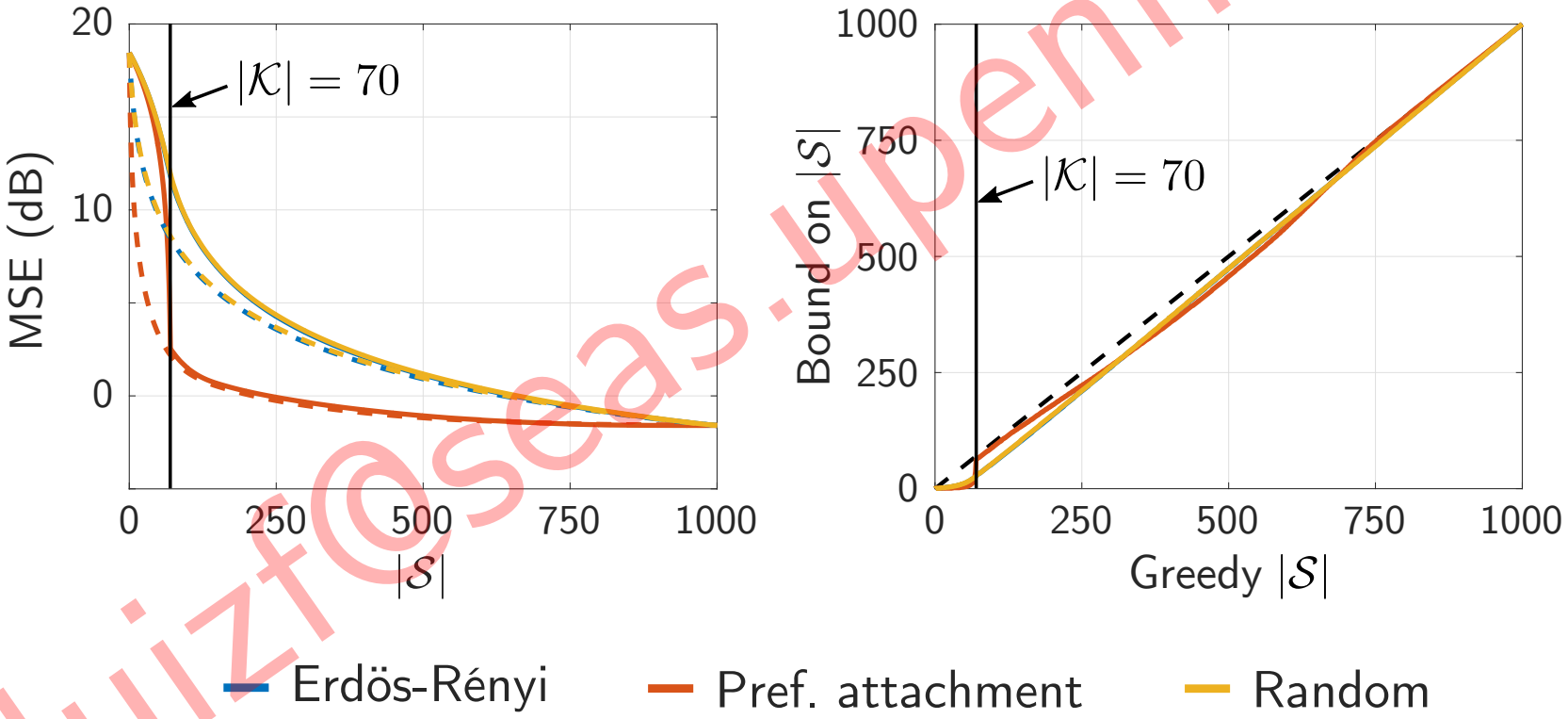
$\mathcal{G}_j = \mathcal{G}_{j-1} \cup \{u\}$

end

end

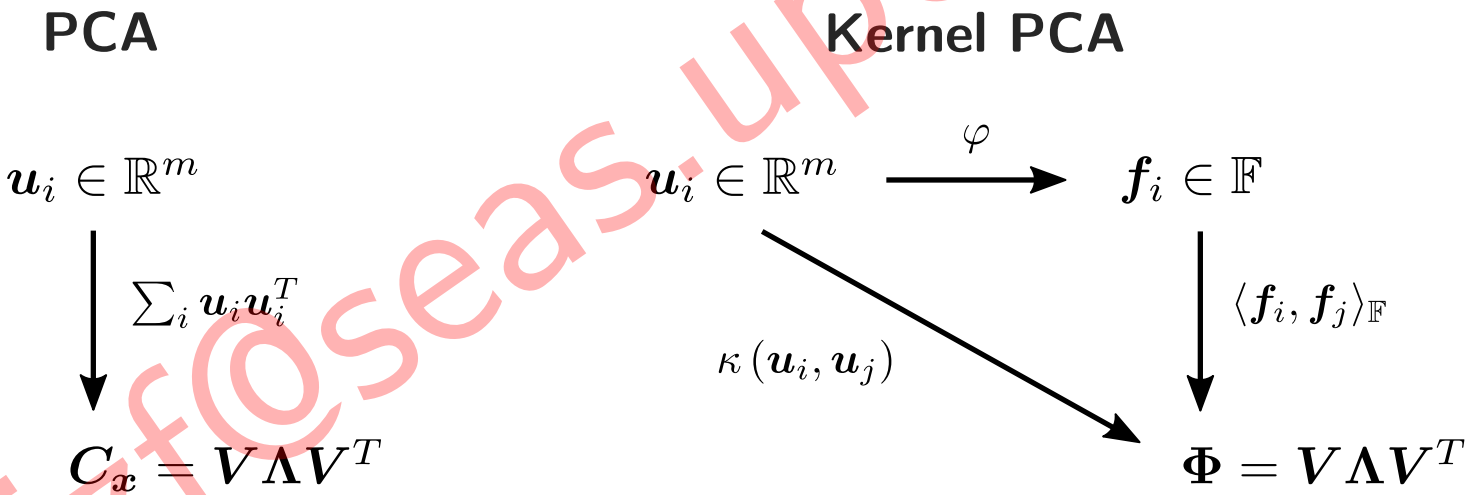
Universal performance bound

- $n = 1000$ nodes, $|\mathcal{K}| = 70$, and SNR $= 20$ dB

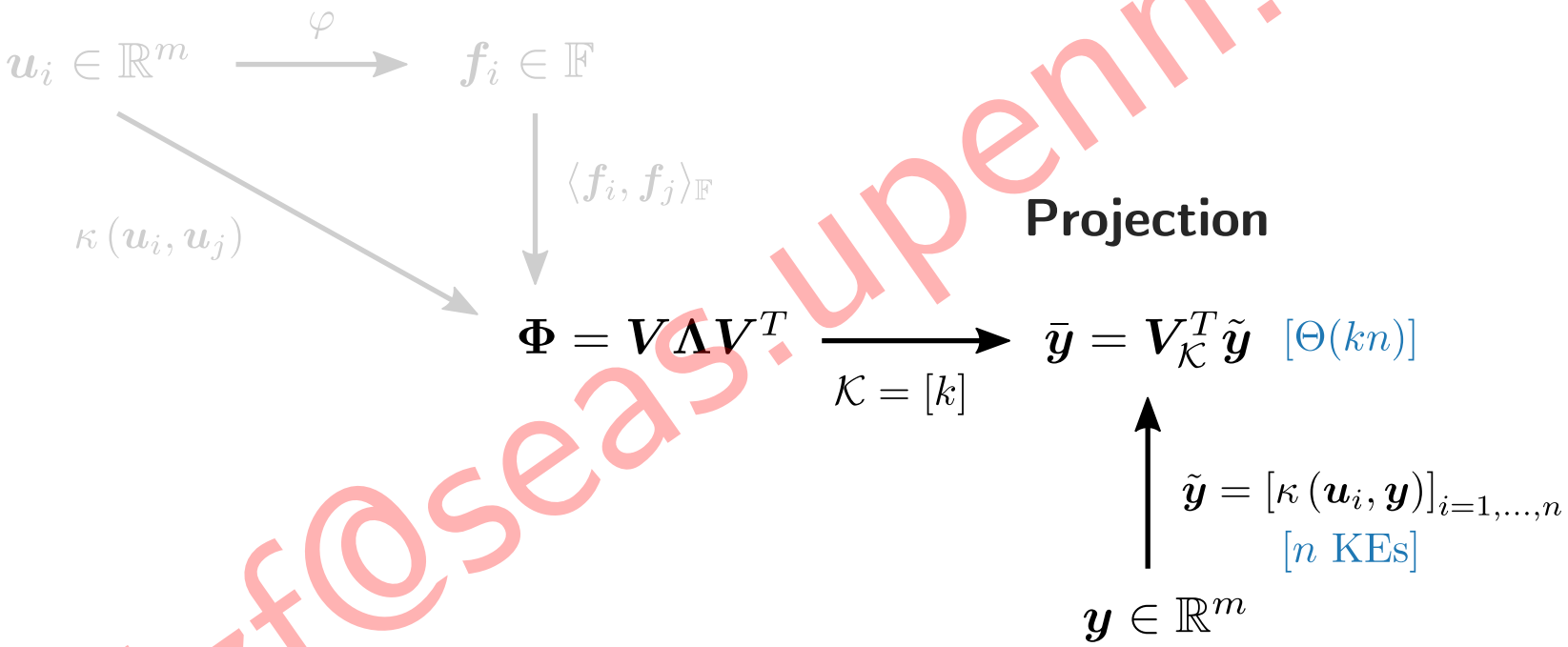


Kernel PCA

- Kernel PCA is a nonlinear version of PCA



Subspace identification



Sampled kernel PCA

Subspace identification

Projection

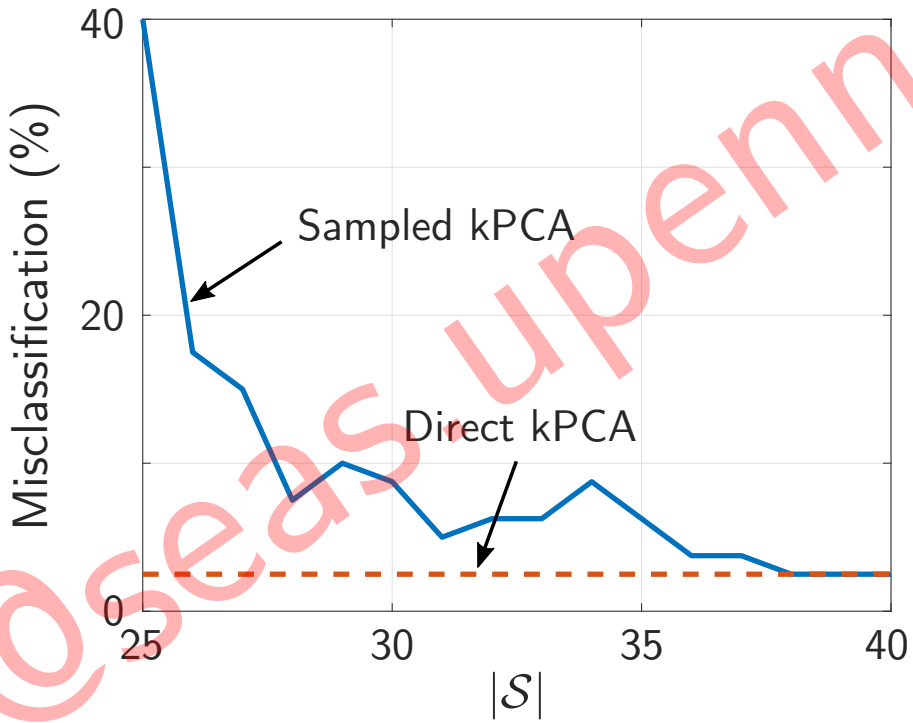
$$\boldsymbol{\Phi} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^T \xrightarrow[\mathcal{K}=[k]]{} \hat{\boldsymbol{y}} = \underbrace{\boldsymbol{V}_{\mathcal{K}}^T \boldsymbol{L}^\star}_{\boldsymbol{P}} \tilde{\boldsymbol{y}}_{\mathcal{S}} \quad [\Theta(k|\mathcal{S}|)]$$

$$\tilde{\boldsymbol{y}}_{\mathcal{S}} = [\kappa(\boldsymbol{u}_i, \boldsymbol{y})]_{i \in \mathcal{S}} \quad [|\mathcal{S}| \text{ KEs}]$$

$$\boldsymbol{y} \in \mathbb{R}^m$$

Sampled kernel PCA

- Kernel PCA (polynomial kernel, $|\mathcal{K}| = 25$)



  - Direct kPCA: 280 KEs and $\mathcal{O}(7000)$ operations
  - Sampled kPCA: 40 KEs and $\mathcal{O}(1000)$ operations

Conclusion

- Sampling graph signals is not straightforward due to the irregularity of the domain
- There are universal performance bounds that can be used to benchmark sampling sets
- Sampling is an effective complexity reduction technique (e.g., sampled kPCA)

Universal Bounds for the Sampling of Graph Signals

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More details: `http://www.seas.upenn.edu/~luizf`