DETERMINATION OF THE MINIMUM DISTANCE BETWEEN SYMBOLS OF THE TWO NON-ORTHOGONAL M-QAM CARRIERS

Signal Processing Lab
University of São Paulo

Luiz F. O. Chamon
chamon@usp.br

Antonio Macílio Pereira de Lucena
macilio@roen.inpe.br

UNIFOR

ABSTRACT

Multicarrier communication systems have become ubiquitous, mainly due to the popularization of OFDM in which carriers are separated in frequency by the inverse of the symbol duration. Recently, more spectrally efficient modulations based on non-orthogonal carriers (non-OFDM) have been put forward and shown numerically to have the same performance as OFDM employing up to 40% less bandwidth. This work addresses the issue of analytically deriving the minimum frequency separation which does not affect the minimum distance between multicarrier symbols. In doing so, it shows that the probability of error remains unaffected up to a certain degree of spectral superposition of the carriers, so that the BER of non-OFDM remains the same as that of OFDM. Simulations and comparisons to previous numerical results are used to illustrate this conclusion.

INTRODUCTION

OFDM: $\Delta f T = 1$

non-OFDM: $\Delta f T < 1$

PROBLEM FORMULATION

Single carrier communication

$\mathcal{C} = \{x_m\},\ x_m \in \mathbb{C} \to M$-symbols constellation

$s_m(t) = \operatorname{Re}\{x_m g(t) e^{j2\pi f_0 t}\} \to$ band pass signal of x_m

$g(t) \to$ pulse shape

Multicarrier communication

$\boldsymbol{x}_\ell \in \mathcal{C}^N \to N \times 1$ multicarrier symbol (N carriers)

$s(t) = \sum_k \operatorname{Re}\{\boldsymbol{x}(k)^T \boldsymbol{\psi}(t-kT) g(t-kT)\} \to$ band pass signal of a sequence $\{\boldsymbol{x}(k)\}$ of multicarrier symbols

$\boldsymbol{\psi}(t) = [\ e^{j2\pi f_0 t}\ \ \cdots\ \ e^{j2\pi[f_0+(N-1)\Delta f]t}\ ]^T \to$ carriers vector

$\Delta f \to$ carriers spectral separation

AWGN channel

Received signal: $r(t) = s(t) + v(t)$

$v(t) \to$ white, zero mean, Gaussian, PSD $= \dfrac{N_0}{2}$

MLE: $\min\limits_{s_\ell} \mathcal{D}_k[r(t), s_\ell(t)] = \displaystyle\int_0^T |r(kT+\tau) - s_\ell(\tau)|^2\, d\tau$

Probability of error

$$P[e_j(k)] = 1 - \Phi\left(\sqrt{\frac{\mathcal{D}_k[s_i(t), s_j(t)]}{2N_0}}\right)$$

Single carrier $(x_i, x_j \in \mathcal{C})$

$$\mathcal{D}_k[s_i(t), s_j(t)] = d_{ij}^2 = |x_j - x_i|^2$$

Multicarrier $(\boldsymbol{x}_i, \boldsymbol{x}_j \in \mathcal{C}^N)$

$$\mathcal{D}_k[s_i(t), s_j(t)] = D_{ij}^2(k) = [\boldsymbol{x}_j - \boldsymbol{x}_i]^* \boldsymbol{H}(k) [\boldsymbol{x}_j - \boldsymbol{x}_i]$$

$$\boldsymbol{H}(k) = \begin{bmatrix} 1 & h_1(k) & \cdots & h_{N-1}(k) \\ h_1^*(k) & 1 & \cdots & h_{N-2}(k) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1}^*(k) & h_{N-2}^*(k) & \cdots & 1 \end{bmatrix}$$

$$h_n(k) = \operatorname{sinc}(n\Delta f T) e^{-jn\phi(k)}, \quad \phi(k) = \pi \Delta f T(2k+1)$$

SPECTRAL SEPARATION LOWER BOUND

Theorem 1. *In a multicarrier system composed of $N = 2$ carriers spectrally separated by Δf transmitting symbols from a constellation $\mathcal{C}$ using rectangular-shaped pulse of length T,*

$$\min_{i \neq j} D_{ij}^2(k) = \min_{i \neq j} d_{ij}^2 = d_{min}^2, \forall k \Leftrightarrow \operatorname{sinc}(\Delta f T) \leq 0.5,$$

or using a Taylor series approximation, $\Delta f T > 0.6033$.

$$\boldsymbol{\delta}_{ij} = [\ \delta_{ij,1}\ \ \delta_{ij,2}\ ]^T = \boldsymbol{x}_j - \boldsymbol{x}_i \to \text{difference vector}$$
$$(\delta_{ij,n} = |\delta_{ij,n}|\, e^{j\theta_{ij,n}})$$

$$D_{ij}^2(k) = [\boldsymbol{x}_j - \boldsymbol{x}_i]^* \boldsymbol{H}(k) [\boldsymbol{x}_j - \boldsymbol{x}_i] \qquad (N = 2)$$
$$= \|\boldsymbol{\delta}\|^2 + 2|\delta_1||\delta_2| \operatorname{sinc}(\Delta f T) \cos[\theta_1 - \theta_2 + \phi(k)]$$

LEMMA 1 (*Identical symbols case*)

Lemma 1. *When $\boldsymbol{\delta}$ has a vanishing element,*

$$\min_{i \neq j} D_{ij}^2(k) = d_{min}^2, \forall k.$$

Proof. For $\delta_1 = 0$ or $\delta_2 = 0$, $D_{ij}^2(k) = \|\boldsymbol{\delta}\|^2, \forall k$. Assume $\delta_1 = 0$,

$$\min_{i \neq j} D_{ij}^2(k) = \min_{\delta_2 \neq 0} \|\delta_2\|^2 = d_{min}^2. \qquad \square$$

LEMMA 2 (*Distinct symbols case*)

Lemma 2. *For $\delta_n \neq 0$, $n = 1, 2$, $\max_{\boldsymbol{\delta}} \mathcal{K} = -1/2$.*

Proof. Assuming $\delta_n \neq 0$,

$$\min_{i \neq j} D_{ij}^2(k) \geq d_{min}^2 \Leftrightarrow \cos[\theta_1 - \theta_2 + \phi(k)] \geq \frac{\mathcal{K}}{\operatorname{sinc}(\Delta f T)}$$

$$\mathcal{K} = \frac{d_{min}^2 - \|\boldsymbol{\delta}\|^2}{2|\delta_1||\delta_2|} < 0, \quad \min_{\delta_n \neq 0} \|\boldsymbol{\delta}\|^2 > d_{min}^2.$$

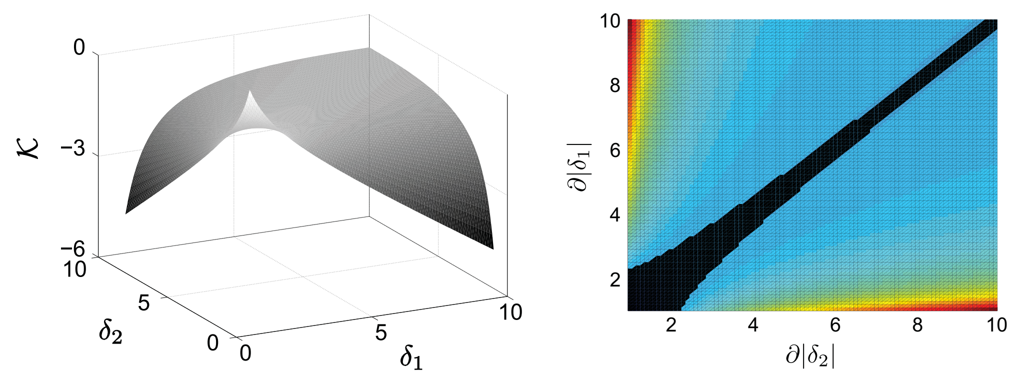
$$\frac{\partial \mathcal{K}}{\partial |\delta_k|} = 0 \Leftrightarrow |\delta_\ell|^2 - |\delta_k|^2 = d_{min}^2, \quad k, \ell = 1, 2$$

(i) $\dfrac{\partial \mathcal{K}}{\partial |\delta_1|} \gtrless 0$ and $\dfrac{\partial \mathcal{K}}{\partial |\delta_2|} \lessgtr 0$ (outside the hyperbolae):

$$\max \mathcal{K} = \lim_{(|\delta_\ell|^2 = d_{min}^2 + |\delta_k|^2, |\delta_k| \to \infty)} \mathcal{K}$$
$$= \lim_{|\delta_k| \to \infty} -\frac{|\delta_k|}{\sqrt{|\delta_k|^2 + d_{min}^2}} = -1, \quad k, \ell = 1, 2$$

(ii) $\dfrac{\partial \mathcal{K}}{\partial |\delta_1|}, \dfrac{\partial \mathcal{K}}{\partial |\delta_2|} < 0$ (inside the hyperbolae):

$$|\delta_n|^2 \geq d_{min}^2 \Rightarrow \max \mathcal{K} = \mathcal{K}\Big|_{|\delta_n|^2 = d_{min}^2} = -\frac{1}{2} \qquad \square$$



LEMMA 3 (Sufficiency lemma)

Lemma 3. *Assuming $\Delta f T \in \mathbb{Q}$ and for some $\epsilon \to 0$,*

$$\min_k \cos[\theta_1 - \theta_2 + \pi(\Delta f T + \epsilon)(2k-1)] = -1.$$

Proof. Constructing ϵ with $\alpha \in \mathbb{Z}$ and $\beta = \begin{cases} 1, & P \text{ is even} \\ 0, & \text{otherwise} \end{cases}$

(i) $\Delta f T = \frac{P}{O}$, *O an odd number*:

$$\epsilon = -\frac{\beta\pi + \theta_1 - \theta_2}{(2\alpha+1)O\pi}$$
$$2k - 1 = (2\alpha+1)O \Rightarrow \cos[(2\alpha+1)P\pi + \beta\pi] = -1$$
$$\lim_{\alpha \to \infty} \epsilon = 0$$

(ii) $\Delta f T = \frac{P}{E}$, *E an even number*:

$$\epsilon = \frac{P}{E(E\alpha-1)} - \frac{\beta\pi + \theta_1 - \theta_2}{\pi(E\alpha-1)} s$$
$$2k - 1 = E\alpha - 1 \Rightarrow \cos(\alpha P\pi + \beta\pi) = -1$$
$$\lim_{\alpha \to \infty} \epsilon = 0$$

PROOF OF THEOREM 1

Proof of Theorem 1. From Lemma 1, $\min D_{ij}^2(k) \leq d_{min}^2, \forall k$.

For $\delta_n \neq 0$,

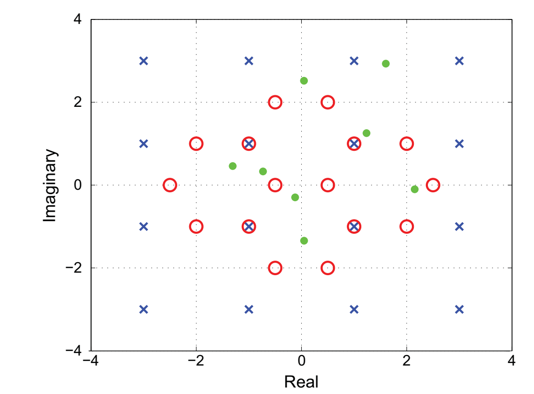
$$\min_{i \neq j} D_{ij}^2(k) \geq d_{min}^2 \Leftrightarrow \cos[\theta_1 - \theta_2 + \phi(k)] \geq \frac{\mathcal{K}}{\operatorname{sinc}(\Delta f T)},$$

and since $\cos(x) \geq -1$,

$$\min_{i \neq j} D_{ij}^2(k) \geq d_{min}^2 \Leftarrow \frac{\mathcal{K}}{\operatorname{sinc}(\Delta f T)} \leq -1 \overset{\text{Lemma 2}}{\Leftarrow} \operatorname{sinc}(\Delta f T) \leq 0.5$$

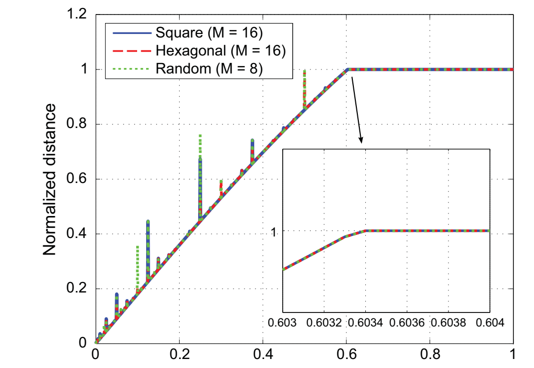
Moreover, Lemma 3 guarantees that close to any $\Delta f T$ there is a $\Delta f T'$ for which $\cos = -1$. Since $\cos$ and sinc are *smooth*, Theorem 1 is an infinitesimally tight bound on sufficiency ($\Rightarrow$). $\square$

SIMULATIONS



Constellations $\mathcal{C}$

- × Square ($M = 16$)
- ○ Hexagonal ($M = 16$)
- • Random ($M = 8$)

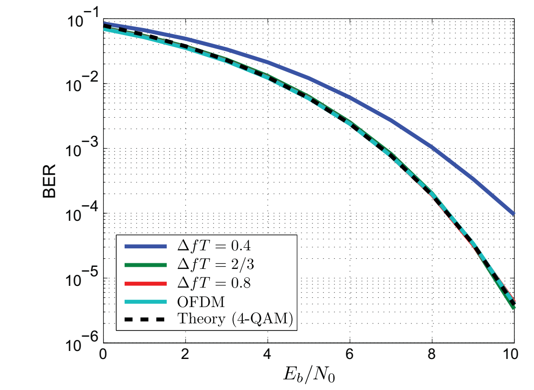


Minimum distance $\times\ \Delta f T$

$N = 2$

$\dfrac{\min D_{ij}^2}{d_{min}^2}$

Theorem 1 $\to \Delta f T > 0.6033\ldots$



BER for 4-QAM