L4DC tutorial
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supervised and reinforcement learning under requirements

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Agenda

I. Constrained supervised learning
   - Constrained learning theory
   - Resilient constrained learning
   - Robust learning

Break (30 min)

II. Constrained reinforcement learning
   - Constrained RL duality
   - Constrained RL algorithms

https://luizchamon.com/l4dc

Why learning under requirements?

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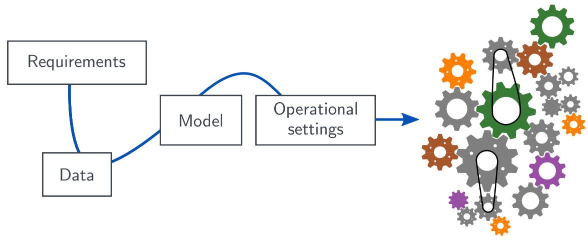
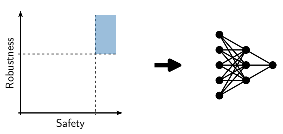


What is a requirements?

- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide



[NASA. "Systems engineering handbook," 2019]

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- **Requirements are "shall" statements:** describe *necessary* features subject to verification
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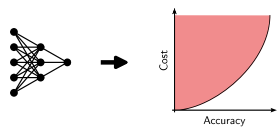
- **Goals are "should" statements:** express recommendations (once "shall" statements are satisfied)
  - *Objective space*: things the system achieves



[NASA. "Systems engineering handbook," 2019]

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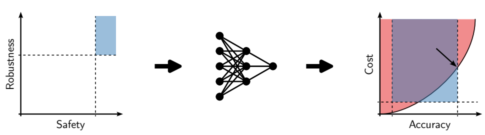

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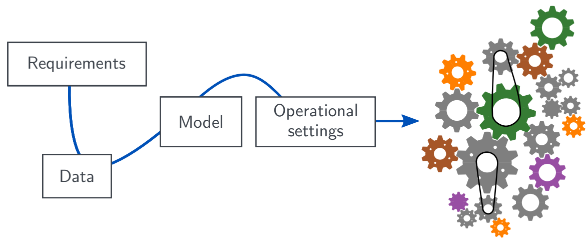

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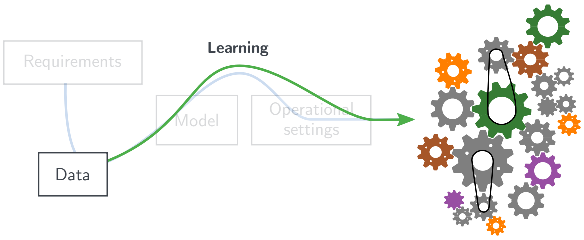

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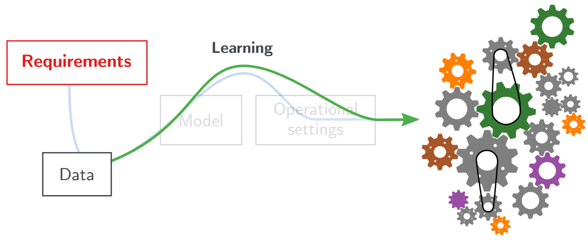

What is (un)constrained learning?

$$P_{\text{U}}^\star = \min_{\theta} \ \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

What is (un)constrained learning?

$$\begin{aligned} P^\star = \min_{\theta} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \le c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \le u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

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What about penalties?

$$\begin{aligned} P^\star = \min_{\theta} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \le c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \le u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

$$\updownarrow$$

$$\min_{\theta} \ \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \lambda\,\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{P}}\Big[\mu(\boldsymbol{x},y)h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

What about penalties?

NON-CONVEX

$$\begin{aligned} P^\star = \min_{\theta} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \ & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \le c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \le u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

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- ✗ There may not exist (λ, μ) such that the penalized solution is optimal *and* feasible
- ✗ Even if such (λ, μ) exist, they are not easy to find (hyperparameter search, cross-validation...)
- ✓ Constrained learning yields stronger guarantees, better performance, better trade-offs...

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
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- Wireless resource allocation
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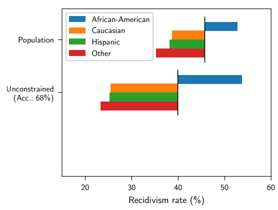
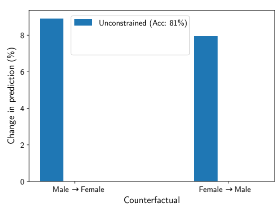
- ...

Fairness

Problem
Predict whether an individual will recidivate



Problem
Predict whether an individual makes > $50k



* We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.

Fairness: “Equality” of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & \text{Prediction error} \\ \text{subject to} \quad & \text{Prediction rate disparity (Race)} \leq c, \\ & \quad \text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\} \end{aligned}$$

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Fairness: “Equality” of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ \text{subject to} \quad & \text{Prediction rate disparity (Race)} \leq c, \\ & \quad \text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\} \end{aligned}$$

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Counterfactual fairness

Problem
Predict whether an individual makes > $50k while being invariant to gender

$$\begin{aligned} \min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ \text{subject to} \quad & \text{Change in prediction } (\rho\boldsymbol{x}) \leq c \quad \text{a.e.} \\ & \quad (\rho : \text{Male} \leftrightarrow \text{Female}) \end{aligned}$$

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[Chamon and Ribeiro, NeurIPS'20]

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- . . .

Federated learning

Problem
Learn a common model using data using data distributed among K clients

$$\min_{\boldsymbol{\theta}} \quad \text{Average loss across clients}$$

- k-th client loss: $\text{Loss}_k(f_{\boldsymbol{\theta}}) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

[Shen et al., ICLR'22]

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Learn a common model using data using data distributed among K clients

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$$\begin{aligned}\min_{\theta} \quad & \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta)\\ \text{subject to} \quad & \text{Loss disparity (k-th client)} \leq c,\\ & k = 1, \dots, K\end{aligned}$$

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Robustness

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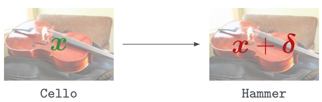
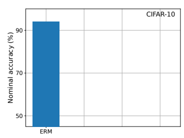
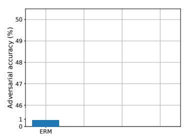
Learn a classifier that is robust to input perturbations







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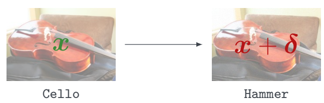

$$\begin{aligned}\min_{\theta} \quad & \text{Nominal accuracy}\\ \text{subject to} \quad & \text{Robustness} \leq c\end{aligned}$$

[Chamon and Ribeiro, NeurIPS'20; Robey*, Chamon*, Pappas, Hassani, and Ribeiro, NeurIPS'21; Chamon, Paternain, Calvo-Fullana, and Ribeiro, IEEE TIT'23]

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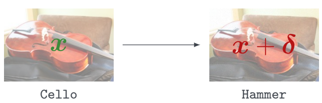

$$\begin{aligned}\min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)\\ \text{subject to} \quad & \text{Robustness} \leq c\end{aligned}$$

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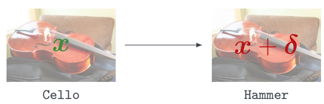

$$\begin{aligned}\min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)\\ \text{subject to} \quad & \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\delta\|_\infty \leq \epsilon} \text{Loss}\big(f_\theta(x_n + \delta), y_n\big)\right] \leq c\end{aligned}$$

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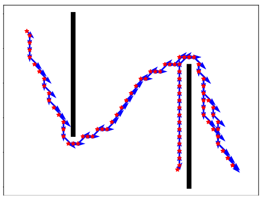
(Manifold) smoothness

Problem
Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories



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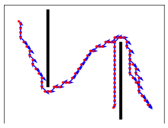

$$\min_{\theta} \quad \text{Imitation error}$$
$$\text{subject to} \quad \text{Smoothness in free space} \leq L$$

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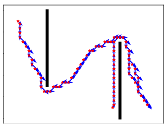

$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), \boldsymbol{u}_n\big)$$
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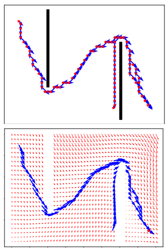

$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), \boldsymbol{u}_n\big)$$
$$\text{subject to} \quad \max_{\boldsymbol{x}\in\mathcal{M}} \|\nabla_{\mathcal{M}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\|^2 \leq L$$

Applications

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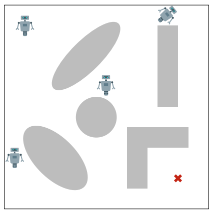
Safety

Problem
Find a control policy that navigates the environment effectively and safely



Safety

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Find a control policy that navigates the environment effectively and safely

$$\underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} \quad \text{Task reward}$$
$$\text{subject to} \quad \Pr\left[\text{Colliding with } \mathcal{O}_i\right] \leq \delta, \quad \text{for } i = 1, 2, \dots$$

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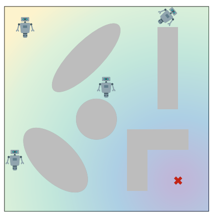

$$\underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} \quad \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t, a_t)\right]$$
$$\text{subject to} \quad \Pr\left[\text{Colliding with } \mathcal{O}_i\right] \leq \delta, \quad \text{for } i = 1, 2, \dots$$

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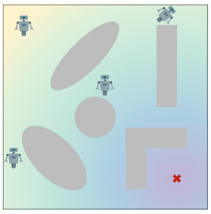

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$$\text{subject to} \quad \Pr\left(\bigcap_{t=0}^{T-1}\{s_t \notin \mathcal{O}_i\} \,\middle|\, \pi\right) \geq 1-\delta_i,$$

$$\text{for } i = 1, 2, \ldots$$

[Paternain et al., IEEE TAC'23]

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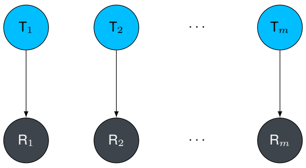
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- ...

Wireless resource allocation

Problem
Allocate the least transmit power to m device pairs to achieve a communication rate



Wireless resource allocation

Problem
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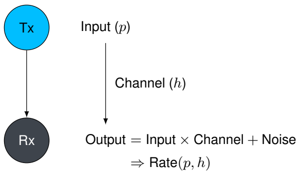

Wireless resource allocation

Problem
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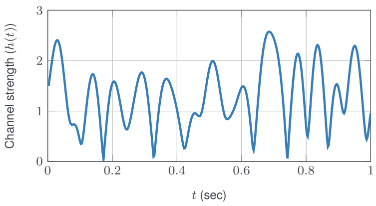

Wireless resource allocation

Problem
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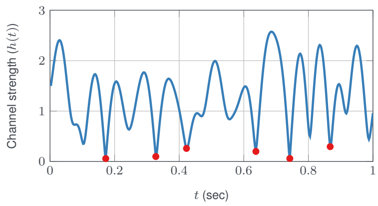

Wireless resource allocation

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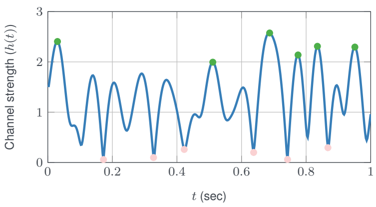

Wireless resource allocation

Problem
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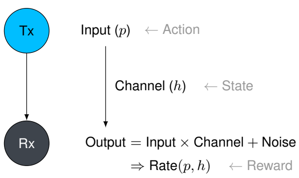

Wireless resource allocation

Problem
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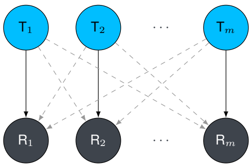
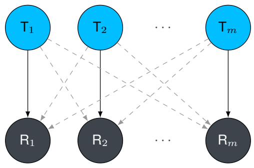


$$\begin{aligned}\min_{\boldsymbol p} &\quad \text{Total transmit power}\\ \text{s. to} &\quad \text{Rate } T_i \to R_i \geq c_i\end{aligned}$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate the least transmit power to m device pairs to achieve a communication rate

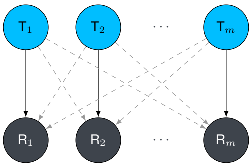


$$\begin{aligned}\min_{\boldsymbol p} &\quad \sum_{i=1}^m \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} p_i(h_t)\right]\\ \text{s. to} &\quad \text{Rate } T_i \to R_i \geq c_i\end{aligned}$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate the least transmit power to m device pairs to achieve a communication rate

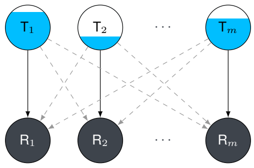


$$\begin{aligned}\min_{\boldsymbol p} &\quad \sum_{i=1}^m \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} p_i(h_t)\right]\\ \text{s. to} &\quad \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} \text{Rate}_i\big(\boldsymbol p(h_t), h_t\big)\right] \geq c_i\end{aligned}$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate the least transmit power to m device pairs to achieve a communication rate

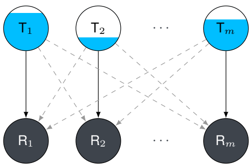


$$\begin{aligned}\min_{\boldsymbol p} &\quad \text{Total probability of depleting battery}\\ \text{s. to} &\quad \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} \text{Rate}_i\big(\boldsymbol p(h_t), h_t\big)\right] \geq c_i\end{aligned}$$

[Chowdhury, Paternain, Verma, Swami, Segarra, Asilomar'23]

Wireless resource allocation

Problem
Allocate the least transmit power to m device pairs to achieve a communication rate



$$\begin{aligned}\min_{\boldsymbol p} &\quad \sum_{i=1}^m \Pr\left[\bigcap_{t=0}^{T-1} \{b_{i,t} = 0\}\right]\\ \text{s. to} &\quad \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} \text{Rate}_i\big(\boldsymbol p(h_t), h_t\big)\right] \geq c_i\end{aligned}$$

[Chowdhury, Paternain, Verma, Swami, Segarra, Asilomar'23]

And many more...

- Precision, recall, churn (e.g., [Cotter et al., JMLR'19])
- Scientific priors (e.g., [Lu et al., SIAM J. Sci. Comp.'21])
- Continual learning (e.g., [Peng et al., ICML'23])
- Active learning (e.g., [Elenter et al., NeurIPS'22])
- Data augmentation (e.g., [Hounie et al., ICML'23])
- Semi-supervised learning (e.g., [Cerviño et al., ICML'23])
- Minimum norm interpolation, SVM...

Constrained supervised learning

What is (un)constrained learning?

$$\begin{aligned}\hat P^\star = \min_{\boldsymbol\theta} &\quad \frac{1}{N}\sum_{n=1}^N \ell\big(f_{\boldsymbol\theta}(\boldsymbol x_n), y_n\big)\\ \text{subject to} &\quad \frac{1}{N}\sum_{m=1}^N g\big(f_{\boldsymbol\theta}(\boldsymbol x_m), y_m\big) \leq c\\ &\quad h\big(f_{\boldsymbol\theta}(\boldsymbol x_r), y_r\big) \leq u, \quad r = 1, \dots, N\end{aligned}$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol\theta}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $(\boldsymbol x_n, y_n) \sim \mathfrak{D}$, $(\boldsymbol x_m, y_m) \sim \mathfrak{A}$, $(\boldsymbol x_r, y_r) \sim \mathfrak{P}$ (i.i.d.)

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

What is (un)constrained learning?

$$\begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{D}}\Big[\ell\big(f_\theta(x), y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{A}}\Big[g\big(f_\theta(x), y\big)\Big] \leq c \\ & h\big(f_\theta(x), y\big) \leq u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- f_θ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \leq c \\ & h\big(f_\theta(x_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{D}}\Big[\ell\big(f_\theta(x), y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{A}}\Big[g\big(f_\theta(x), y\big)\Big] \leq c \\ & h\big(f_\theta(x), y\big) \leq u \text{ a.e.} \end{aligned}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \leq c \\ & h\big(f_\theta(x_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{D}}\Big[\ell\big(f_\theta(x), y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{A}}\Big[g\big(f_\theta(x), y\big)\Big] \leq c \\ & h\big(f_\theta(x), y\big) \leq u \text{ a.e.} \end{aligned}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \leq c \\ & h\big(f_\theta(x_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{D}}\Big[\ell\big(f_\theta(x), y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{A}}\Big[g\big(f_\theta(x), y\big)\Big] \leq c \\ & h\big(f_\theta(x), y\big) \leq u \text{ a.e.} \end{aligned}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\theta} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \leq c \\ & h\big(f_\theta(x_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{D}}\Big[\ell\big(f_\theta(x), y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(x,y)\sim\mathfrak{A}}\Big[g\big(f_\theta(x), y\big)\Big] \leq c \\ & h\big(f_\theta(x), y\big) \leq u \text{ a.e.} \end{aligned}$$

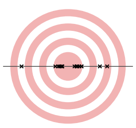
Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

What classical learning theory says?

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big) \ \xrightarrow{\text{"LLN"}} \ \min_{\theta} \ \mathbb{E}\Big[\text{Loss}\big(f_\theta(x), y\big)\Big]$$

✔ f_θ is *probably approximately correct (PAC)* learnable
e.g., linear functions, smooth functions (finite RKHS norm, bandlimited), NNs…
$(N \approx 1/\epsilon^2)$



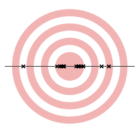
[Rostamizadeh, Talwalkar, Mohri. Foundations of machine learning, 2012]; [Ben-David, Shalev-Shwartz. Understanding machine learning…, 2014]

What classical learning theory says?

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big) \ \xrightarrow{\text{"LLN"}} \ \min_{\theta} \ \mathbb{E}\Big[\text{Loss}\big(f_\theta(x), y\big)\Big]$$

✔ f_θ is *probably approximately correct (PAC)* learnable
e.g., linear functions, smooth functions (finite RKHS norm, bandlimited), NNs…
$(N \approx 1/\epsilon^2)$

✘ Requirements?



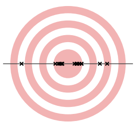
[Rostamizadeh, Talwalkar, Mohri. Foundations of machine learning, 2012]; [Ben-David, Shalev-Shwartz. Understanding machine learning…, 2014]

What's in a solution?

Definition (PAC learnability)

f_{θ} is a *probably approximately correct* (PAC) learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\theta^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal $$P^\star - \mathbb{E}_{(x,y)\sim\mathfrak{D}}\left[\ell\left(f_{\theta^\dagger}(x), y\right)\right] \le \epsilon$$

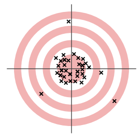


What's in a solution?

Definition (PACC learnability)

f_{θ} is a *probably approximately correct constrained* (PACC) learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\theta^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal $$\left|P^\star - \mathbb{E}_{(x,y)\sim\mathfrak{D}}\left[\ell\left(f_{\theta^\dagger}(x), y\right)\right]\right| \le \epsilon$$
- approximately feasible $$\mathbb{E}_{(x,y)\sim\mathfrak{A}}\left[g\left(f_{\theta^\dagger}(x), y\right)\right] \le c + \epsilon$$



When is constrained learning possible?

$$\hat{P}^\star = \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_\theta(x_n), y_n\right) \quad \text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\left(f_\theta(x_m), y_m\right) \le c$$

$\xrightarrow{?}$

$$P^\star = \min_{\theta} \mathbb{E}_{(x,y)\sim\mathfrak{D}}\left[\ell\left(f_\theta(x), y\right)\right] \quad \text{subject to} \quad \mathbb{E}_{(x,y)\sim\mathfrak{A}}\left[g\left(f_\theta(x), y\right)\right] \le c$$

Proposition

f_θ is PAC learnable $\not\Rightarrow$ f_θ is PACC learnable

ECRM is not a PACC learner

Counter-example

$$P^\star = \min_{\theta\in\Theta} J(\theta) \quad \text{subject to} \quad \theta_2\,\mathbb{E}_\tau[\tau] \le \theta_1 - 1, \quad -\theta_1\,\mathbb{E}_\tau[\tau] \le \theta_2 - 1$$

$$J(\theta) = \begin{cases} 1/16, & \theta = [1/2, 1/2] \\ 1/8, & \theta = [1, 1] \\ 1/4, & \theta = [1, 0] \end{cases}$$

- $\tau \sim \text{Uniform}\left(-1/2, 1/2\right)$

ECRM is not a PACC learner

Counter-example

$$P^\star = \min_{\theta\in\Theta} J(\theta) = \frac{1}{8} \quad \text{subject to} \quad \theta_2\,\mathbb{E}_\tau[\tau] \le \theta_1 - 1 \Rightarrow \theta_1 \ge 1, \quad -\theta_1\,\mathbb{E}_\tau[\tau] \le \theta_2 - 1 \Rightarrow \theta_2 \le 1$$

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$$J(\theta) = \begin{cases} 1/16, & \theta = [1/2, 1/2] \\ 1/8, & \theta = [1, 1] \\ 1/4, & \theta = [1, 0] \end{cases}$$

$$\hat{P}^\star = \min_{\theta\in\Theta} J(\theta) \quad \text{subject to} \quad \theta_2\bar{\tau}_N \le \theta_1 - 1, \quad -\theta_1\bar{\tau}_N \le 1 - \theta_2$$

$$\Pr\left[|\hat{P}^\star - P^\star| \le 1/32\right] = \Pr\left[\bar{\tau}_N = 0\right] = 0$$

- $\tau \sim \text{Uniform}\left(-1/2, 1/2\right) \to \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^{N}\tau_n$

ECRM is not a PACC learner

Counter-example

$$P^\star = \min_{\theta\in\Theta} J(\theta) = \frac{1}{8} \quad \text{subject to} \quad \theta_2\,\mathbb{E}_\tau[\tau] \le \theta_1 - 1 \Rightarrow \theta_1 \ge 1, \quad -\theta_1\,\mathbb{E}_\tau[\tau] \le \theta_2 - 1 \Rightarrow \theta_2 \le 1$$

$$J(\theta) = \begin{cases} 1/16, & \theta = [1/2, 1/2] \\ 1/8, & \theta = [1, 1] \\ 1/4, & \theta = [1, 0] \end{cases}$$

$$\hat{P}_r^\star = \min_{\theta\in\Theta} J(\theta) \quad \text{subject to} \quad \theta_2\bar{\tau}_N \le \theta_1 - 1 + r_1, \quad -\theta_1\bar{\tau}_N \le 1 - \theta_2 + r_2$$

$$\Pr\left[|\hat{P}_r^\star - P^\star| \le 1/32\right] \le 4e^{-0.001N},$$

unless $\bar{\tau}_N \le r_1 < \frac{\bar{\tau}_N + 1}{2}$ and $r_2 \ge \bar{\tau}_N$

- $\tau \sim \text{Uniform}\left(-1/2, 1/2\right) \to \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^{N}\tau_n$

Constrained learning challenges

$$\hat{P}^\star = \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_\theta(x_n), y_n\right) \quad \text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\left(f_\theta(x_m), y_m\right) \le c, \quad h\left(f_\theta(x_r), y_r\right) \le u$$

PAC $\not\rightarrow$

$$P^\star = \min_{\theta} \mathbb{E}_{(x,y)\sim\mathfrak{D}}\left[\ell\left(f_\theta(x), y\right)\right] \quad \text{subject to} \quad \mathbb{E}_{(x,y)\sim\mathfrak{A}}\left[g\left(f_\theta(x), y\right)\right] \le c, \quad h\left(f_\theta(x), y\right) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?

2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c \quad \xrightarrow{\text{PAC}} \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r, y_r\big) \le u \qquad h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?

2) *Computational*: can we solve the constrained empirical problem?

Duality

PRIMAL

↕

DUAL

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

↕

DUAL

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

↕

$$\hat{D}^\star = \max_{\lambda \ge 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

↕

$$\hat{D}^\star = \max_{\lambda \ge 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

- In general, $\hat{D}^\star \le \hat{P}^\star$
- But in some cases, $\hat{D}^\star = \hat{P}^\star$ (strong duality) [e.g., convex optimization]

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

$\not\updownarrow$

$$\hat{D}^\star = \max_{\lambda \ge 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

- In general, $\hat{D}^\star \le \hat{P}^\star$
- ~~But in some cases, $\hat{D}^\star = \hat{P}^\star$ (strong duality) [e.g., convex optimization]~~

An alternative path

$$\hat{P}^\star = \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to} \quad \frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \le c$$

PRIMAL $\not\leftrightarrow$ DUAL

$$\hat{D}^\star = \max_{\lambda \ge 0} \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\left(\frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\right)$$

$\not\updownarrow$ PAC

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[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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$\downarrow \mathcal{H}_\theta \subset \mathcal{H}$

$$\tilde{P}^\star = \min_{\phi\in\mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\big[\ell(\phi, \boldsymbol{z})\big] \quad \text{s. to} \quad \mathbb{E}_{\boldsymbol{z}}\big[g(\phi, \boldsymbol{z})\big] \le c \quad \overset{?}{\longleftrightarrow} \quad \tilde{D}^\star = \max_{\lambda \ge 0} \min_{\phi\in\mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\big[\ell(\phi, \boldsymbol{z})\big] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\big[g(\phi, \boldsymbol{z})\big] - c\right)$$

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Non-convex variational duality

Convex optimization: Primal ⟷ Dual

Non-convex, finite dimensional optimization: Primal ⟷̸ Dual

Non-convex variational duality

Convex optimization: Primal ⟷ Dual

Non-convex, finite dimensional optimization: Primal ⟷̸ Dual

Non-convex, **infinite** dimensional optimization: Primal ⟷ Dual

[Chamon et al., IEEE TSP'20]

Sparse logistic regression

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \boldsymbol{\theta}^T \boldsymbol{x}_n\right)\right]$$
$$\text{s. to } \|\boldsymbol{\theta}\|_0 = \sum_{t=1}^{p} \mathbb{I}\left[\boldsymbol{\theta}_t \neq 0\right] \leq k$$

Discrete, non-convex

[Chen et al., JMLR'19]: NP-hard

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$$\min_{\theta \in L_2} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \int \theta(t) x_n(t) dt\right)\right]$$
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Continuous, non-convex

[Chamon et al., IEEE TSP'20]: tractable

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⟷̸

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$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda \left(\frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c \right)$$

↕̸ PAC

$$P^\star = \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \leq c$$

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↕ ϵ_0 ↕ ϵ_0

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↕ $O(\epsilon)$

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[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

Dual (near-)PACC learning

Theorem

Let f be ν-universal, i.e., for each $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\gamma \in [0,1]$ there exists $\boldsymbol{\theta}$ such that

$$\mathbb{E}\left[\left|\gamma f_{\boldsymbol{\theta}_1}(\boldsymbol{x}) + (1-\gamma) f_{\boldsymbol{\theta}_2}(\boldsymbol{x}) - f_{\boldsymbol{\theta}}(\boldsymbol{x})\right|\right] \le \nu$$

[$\{f_\theta\}$ is a good covering of $\overline{\text{conv}}(\{f_\theta\})$]

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Then $\hat{D}^\star$ is a (near-)PACC learner, i.e., there exists a solution $\boldsymbol{\theta}^\dagger$ that, with probability $1-\delta$,

Near-optimal: $\left|P^\star - \hat{D}^\star\right| \le \widetilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

Approximately feasible: $\mathbb{E}\left[g\left(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\right)\right] \le c + \widetilde{O}\left(\frac{1}{\sqrt{N}}\right)$

(mild conditions apply)

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(if losses are convex) $h\left(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\right) \le r$, with $\mathfrak{P}$-prob. $1 - \widetilde{O}\left(\frac{1}{\sqrt{N}}\right)$

(mild conditions apply)

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Theorem

Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$. There exists $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ achieving $\hat{D}^\star$ such that $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL), i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\left(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\right)\right] \le c + \epsilon$$

$\epsilon_0 = M\nu$ $\quad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]}$ $\quad \Delta = \max\left(\left\|\boldsymbol{\lambda}^\star\right\|_1, \left\|\hat{\boldsymbol{\lambda}}^\star\right\|_1, \left\|\tilde{\boldsymbol{\lambda}}^\star\right\|_1\right)$

Sources of error

parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

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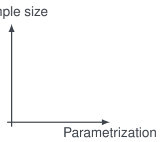
Dual learning trade-offs

- Unconstrained learning

  parametrization × sample size



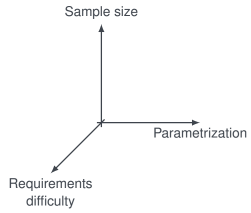
Dual learning trade-offs

- Unconstrained learning

  parametrization × sample size

- Constrained learning

  parametrization × sample size × requirements



When is constrained learning possible?

Corollary

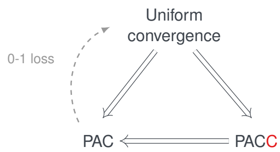
f_θ is PAC learnable $\approx^*$ f_θ is PACC learnable

Constrained learning is **essentially as hard as** unconstrained learning

[mild conditions apply]

When is constrained learning possible?

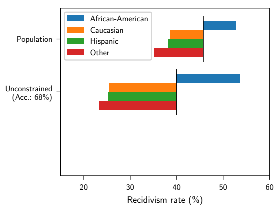
Corollary



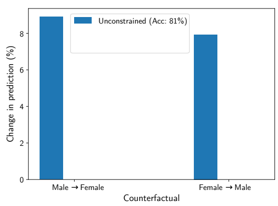
[mild conditions apply]

Fairness

Problem
Predict whether an individual will recidivate



Problem
Predict whether an individual makes > $50k



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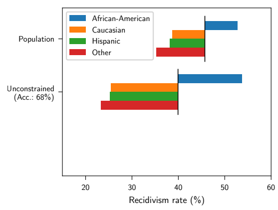

Problem
Predict whether an individual makes > $50k

Fairness: "Equality" of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to } \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[f_\theta(\boldsymbol{x}_n)=1 \mid \text{Race}] \le \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}[f_\theta(\boldsymbol{x}_n)=1] + c,$$

for Race $\in$ {African-American, Caucasian, Hispanic, Other}

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$$\text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\}$$

*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.
[Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23]

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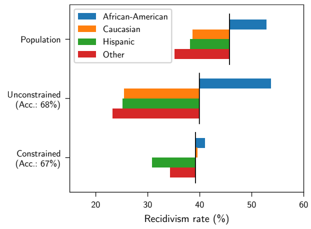

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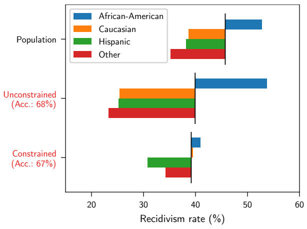

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Fairness: “Equality” of odds

	True label	Unconstrained: Prediction 0	Unconstrained: Prediction 1	Constrained: Prediction 0	Constrained: Prediction 1
African-American	0	31%	16%	36%	11%
	1	16%	37%	23%	30%
Caucasian	0	52%	9%	44%	17%
	1	23%	16%	16%	23%

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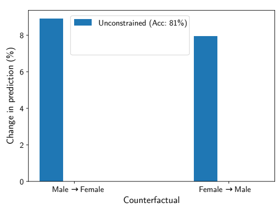

*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.

Counterfactual fairness

Problem
Predict whether an individual makes > $50k while being invariant to gender

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \quad \mathrm{D}_{\mathrm{KL}}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \,\|\, f_{\boldsymbol{\theta}}(\rho\boldsymbol{x}_n)\big) \leq c, \quad \text{for all } n$$

$$(\rho : \text{Male} \leftrightarrow \text{Female})$$

[Chamon and Ribeiro, NeurIPS'20]

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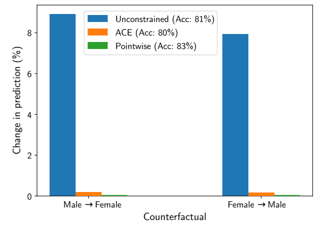

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↕

$$\max_{\lambda_n \geq 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \sum_{n=1}^{N} \lambda_n \left[\mathrm{D}_{\mathrm{KL}}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \| f_{\boldsymbol{\theta}}(\rho\boldsymbol{x}_n)\right) - c\right]$$

[Chamon and Ribeiro, NeurIPS'20]

Counterfactual fairness

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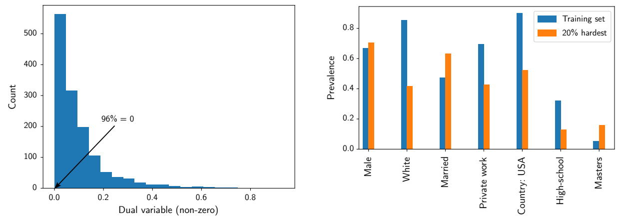

[Chamon and Ribeiro, NeurIPS'20]

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Constrained optimization methods

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \leq u$$

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  e.g., conditional gradients (Frank-Wolfe)
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  e.g., barriers, projection, polyhedral approx.

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- Duality
  e.g., (augmented) Lagrangian
  - ✓ Tractability
  - ✓ (near-)feasible solution [small duality gap]

Dual learning algorithm

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Dual learning algorithm

- Minimize the primal (≡ **ERM**)

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \Big[\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\Big]$$

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[Haeffele et al., CVPR'17; Ge et al., ICLR'18; Mei et al., PNAS'18; Kawaguchi et al., AISTATS'20. . .]

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- Update the dual

$$\lambda^+ = \left[\lambda + \eta\left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}^+}(\boldsymbol{x}_m), y_m\big) - c\right)\right]_+$$

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A (near-)PACC learner

Theorem
Suppose $\boldsymbol{\theta}^\dagger$ is a ρ-approximate solution of the regularized ERM:

$$\boldsymbol{\theta}^\dagger \approx \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \Big(\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\Big).$$

Then, after $T = \left\lceil \frac{\|\boldsymbol{\lambda}^\star\|^2}{2\eta M \nu} \right\rceil + 1$ dual iterations with step size $\eta \leq \frac{2\epsilon}{mB^2}$,

the iterates $\big(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\big)$ are such that

$$\left|P^\star - L\Big(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\Big)\right| \leq (2 + \Delta)(\epsilon_0 + \epsilon) + \rho$$

with probability $1 - \delta$ over sample sets.

[Chamon et al., IEEE TIT'23]

In practice. . .

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In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_{\boldsymbol{\beta}} \left[\ell\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right) + \lambda_{t-1} g\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right) \right]$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N} \sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_m\right) - c \right) \right]_+$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$

SGD

Dual update

PyTorch
https://github.com/lfochamon/csl

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Check slack:
- feasibility: $s_t \leq 0$
- "duality gap": $\lambda_t s_t$

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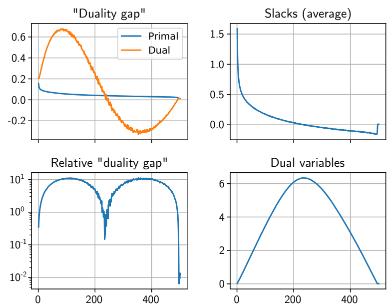

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Penalty-based vs. dual learning

Penalty-based learning

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta}} \; \text{Loss}(\boldsymbol{\theta}) + \lambda \cdot \text{Penalty}(\boldsymbol{\theta})$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss + λPenalty

Dual learning

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$$\lambda^+ = \left[\lambda + \eta \left(\text{Penalty}(\boldsymbol{\theta}^\dagger) - c \right) \right]_+$$

- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\leq c$

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Heterogeneous federated learning

Problem
Learn a common model using data using data distributed among K clients

$$\min_{\boldsymbol{\theta}} \; \frac{1}{K} \sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}})$$
$$\text{subject to} \quad \text{Loss}_k(f_{\boldsymbol{\theta}}) \leq \frac{1}{K} \sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}}) + c, \quad k = 1, \dots, K$$

- k-th client loss: $\text{Loss}_k(\phi) = \frac{1}{N_k} \sum_{n_k=1}^{N_k} \text{Loss}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\right)$

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Resilient constrained learning

Definition (Resilience)
(ecology) ability of an ecosystem to adapt its function to accommodate operating conditions

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Resilient constrained learning

Definition (Resilient equilibrium)

For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \quad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

[Hounie et al., NeurIPS'23]

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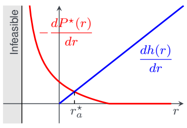

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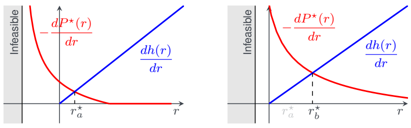

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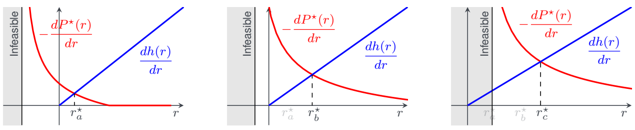

[Hounie et al., NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)

For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) = \boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$$

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- ✔ After relaxing, $\boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$ is *smaller* than $\boldsymbol{\lambda}^\star(\boldsymbol{0})$
  $\Rightarrow$ Resilient constrained learning "generalizes better" (lower sample complexity)

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For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$P^\star(\boldsymbol{r}^\star) = \min_{\boldsymbol{\theta},\boldsymbol{r}} \; \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] + h(\boldsymbol{r})$$
$$\text{subject to} \;\; \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big)\Big] \leq c_i + r_i$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

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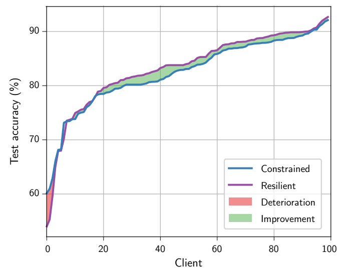
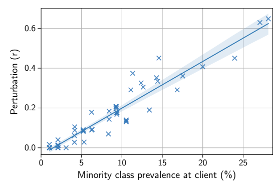
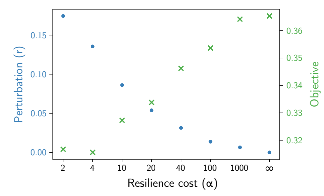
Heterogeneous federated learning



[Hounie et al., NeurIPS'23]

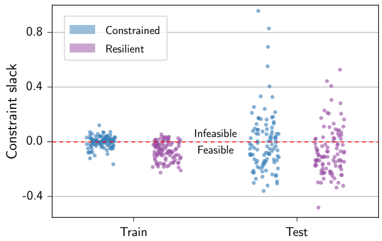
Heterogeneous federated learning





[Hounie et al., NeurIPS'23]

Heterogeneous federated learning



[Hounie et al., NeurIPS'23]

Summary

- **Constrained learning is ~~the~~ a tool to learn under requirements**
- **Constrained learning is hard...**
- **...but possible. How?**

Summary

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  Constrained learning imposes generalizable requirements organically during training, e.g., fairness [Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23], heterogeneity [Shen et al., ICRL'22]...
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Summary

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- **Constrained learning is hard...**
  Constrained, non-convex, statistical optimization problem
- **...but possible. How?**
  We can learn under requirements (essentially) whenever we can learn at all by solving *(penalized) ERM problems.* Resilient learning can then be used to adapt the requirements to the task difficulty [Hounie et al., NeurIPS'23]

Robustness constraints

Agenda

Adversarially robust learning

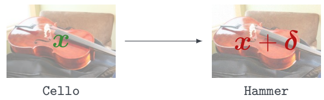
Semi-infinite learning

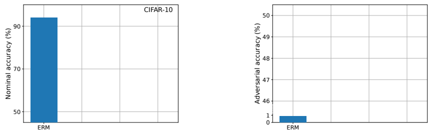
Probabilistic robustness

Robust learning

Problem

Learn an image classifier that is robust to input perturbations





Adversarial training

Problem

Learn an image classifier that is robust to input perturbations

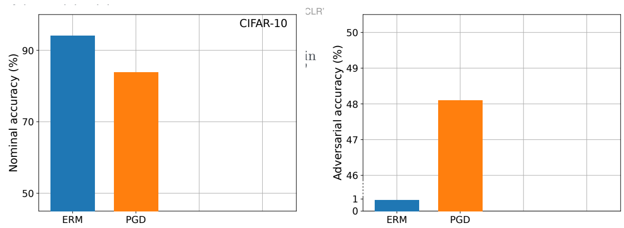
- Adversarial training [Szegedy et al., ICLR'14; Goodfellow et al., ICLR'15; Madry et al., ICLR'18; ...]

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n), y_n\big) \longrightarrow \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$

Adversarial training

Problem

Learn an image classifier that is robust to input perturbations



[Robey et al., NeurIPS'21]

Adversarial training

Problem

Learn an image classifier that is robust to input perturbations

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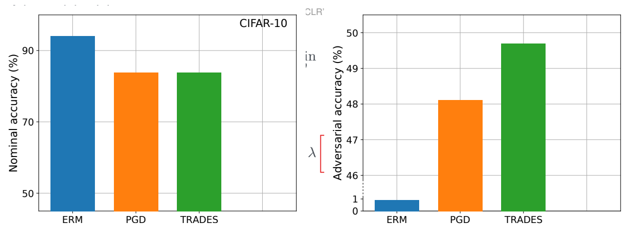
$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n), y_n\big) \qquad \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n), y_n\big) + \lambda\left[\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$

Adversarial training

Problem

Learn an image classifier that is robust to input perturbations



[Zhang et al., ICML'19]

Constrained learning for robustness

Problem

Learn an image classifier that is robust to input perturbations

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n), y_n\big)$$

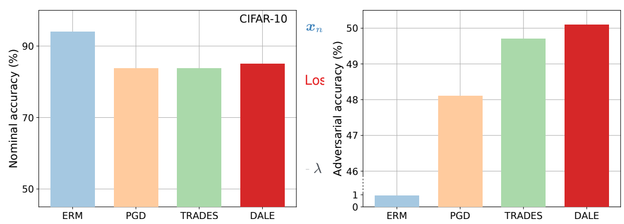
$$\text{subject to } \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\theta}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right] \le c$$

[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Constrained learning for robustness

Problem

Learn an image classifier that is robust to input perturbations



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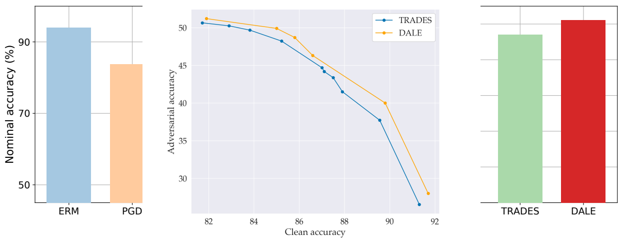
Constrained learning for robustness

Problem
Learn an image classifier th



[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Penalty-based vs. dual learning

Penalty-based learning

$$\theta^\dagger \in \underset{\theta}{\operatorname{argmin}} \; \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss + λPenalty

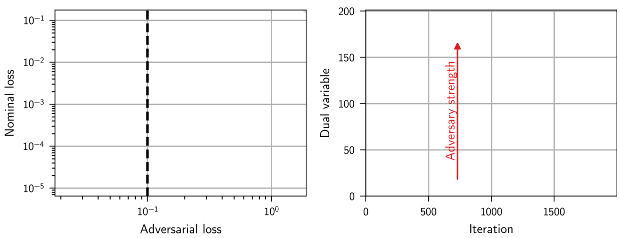
Dual learning

$$\theta^\dagger \in \underset{\theta}{\operatorname{argmin}} \; \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

$$\lambda^+ = \left[\lambda + \eta\left(\text{Penalty}(\theta^\dagger) - c\right)\right]_+$$

- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\leq c$

Constrained learning for robustness



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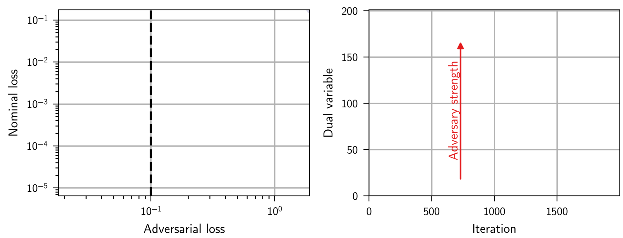

[Chamon et al., IEEE TIT'23]

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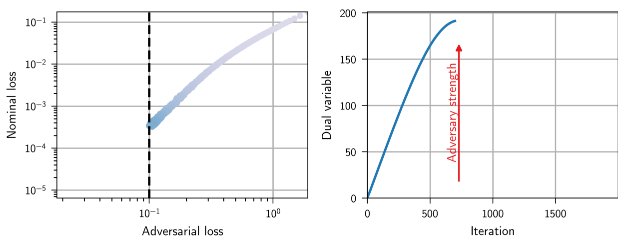

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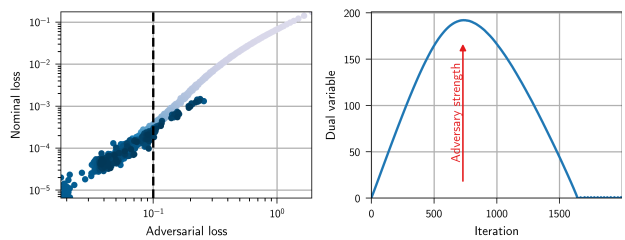
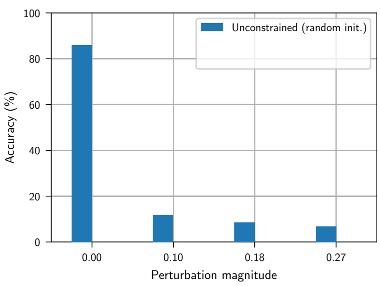
Constrained learning for robustness



Empirical observations: [Zhang et al., ICML'20; Sitawarin, arXiv'20]

Constrained learning for robustness



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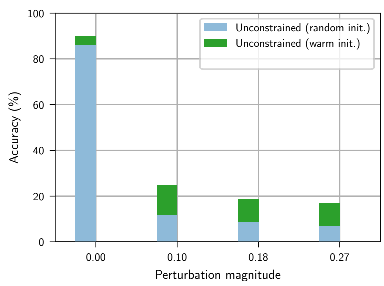

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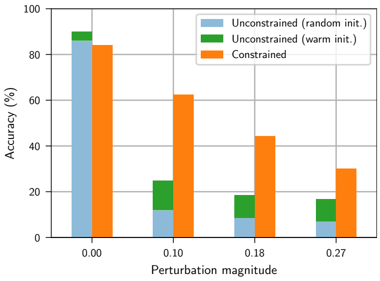

[Chamon et al., IEEE TIT'23]

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda \geq 0} \min_{\theta} \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big) + \lambda \left[\max_{\boldsymbol{\delta} \in \Delta} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

- ✔ Balancing nominal accuracy and robustness ⇒ Dual constrained learning

Constrained learning for robustness

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Learn an image classifier that is robust to input perturbations

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- ✔ Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- ✖ Computing the worst-case perturbations

Adversarial training

$$\min_{\theta} \frac{1}{N} \sum_{n=1}^{N} \left[\max_{\boldsymbol{\delta} \in \Delta} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

- "PGD" [Mądry et al., ICLR'18]

1: $\boldsymbol{\delta}^1 \leftarrow \boldsymbol{\delta}_{t-1}$
2: **for** $k = 1, \dots, K$
3: $\boldsymbol{\delta}^{k+1} \leftarrow \text{proj}_{\Delta}\left[\boldsymbol{\delta}^k + \eta \, \text{sign}\Big(\nabla_{\boldsymbol{\delta}} \text{Loss}\big(f_{\theta^k}(\boldsymbol{x} + \boldsymbol{\delta}^k), y\big)\Big)\right]$
4: **end**
5: $\boldsymbol{\delta}_t \leftarrow \boldsymbol{\delta}^{K+1}$
6: $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \nabla_{\boldsymbol{\theta}} \text{Loss}\Big(f_{\boldsymbol{\theta}}(\boldsymbol{x} + \boldsymbol{\delta}_t), y\Big)$

Adversarial training

$$\min_{\theta} \frac{1}{N} \sum_{n=1}^{N} \left[\max_{\boldsymbol{\delta} \in \Delta} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

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- Random initialization
- Restarts
- Pruning
- Adaptive step size

[Dhillon et al., ICLR'18; Carmon et al., NeurIPS'19; Wu et al., NeurIPS'20; Cheng et al., IJCAI'22]

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda \geq 0} \min_{\theta} \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big) + \lambda \left[\max_{\boldsymbol{\delta} \in \Delta} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

- ✔ Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- ✖ Computing the worst-case perturbations
  - ~~gradient ascent → non-convex, underparametrized~~

Agenda

Adversarially robust learning

Semi-infinite learning

Probabilistic robustness

Semi-infinite constrained learning

$$\min_{\theta} \frac{1}{N} \sum_{n=1}^{N} \left[\max_{\boldsymbol{\delta} \in \Delta} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

Semi-infinite constrained learning

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[t(\boldsymbol{x}_n, y_n)\right]$$
$$\text{subject to} \quad \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big) \le t(\boldsymbol{x}_n, y_n),$$
for all $(\boldsymbol{x}_n, y_n)$ and $\boldsymbol{\delta} \in \Delta$

- Epigraph formulation:
  $\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) \le t \iff \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) \le t$, for all $\|\boldsymbol{\delta}\|_\infty \le \epsilon$

Semi-infinite constrained learning

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[t(\boldsymbol{x}_n, y_n)\right]$$
$$\text{subject to} \quad \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_0), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{\sqrt{2}}), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_e), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_\pi), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_4), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{e^2}), y_n\big) \le t(\boldsymbol{x}_n, y_n)$$
$$\cdots$$

- Epigraph formulation:
  $\max_{\|\boldsymbol{\delta}\|_\infty \le \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) \le t \iff \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) \le t$, for all $\|\boldsymbol{\delta}\|_\infty \le \epsilon$
- Semi-infinite program

Duality

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$
$$\Big\updownarrow =$$
$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[t(\boldsymbol{x}_n, y_n)\right] \ \text{s. to} \ \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big) \le t(\boldsymbol{x}_n, y_n), \ \forall(\boldsymbol{x}_n, y_n, \boldsymbol{\delta})$$
$$\Big\updownarrow =$$
$$\min_{\boldsymbol{\theta}} \ \sup_{\mu\in\mathcal{P}} \ \underbrace{\frac{1}{N}\sum_{n=1}^{N}\int_\Delta \mu_n(\boldsymbol{\delta})\,\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)d\boldsymbol{\delta}}_{L(\boldsymbol{\theta},\mu_n)}$$

Duality

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$
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From optimization to sampling

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$
$$\Big\updownarrow \approx$$
$$\min_{\boldsymbol{\theta}} \ \sup_{\mu\in\mathcal{P}^2} \ \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\boldsymbol{\delta}\sim\mu_\gamma(\cdot|\boldsymbol{x}_n,y_n)}\left[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]}_{L(\boldsymbol{\theta},\mu)}$$

Proposition

For all $\epsilon > 0$, there exists $\gamma(\boldsymbol{x}, y) < \max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big)$ s.t. $L(\boldsymbol{\theta}, \mu_\gamma) \ge \sup_{\mu\in\mathcal{P}^2} L(\boldsymbol{\theta}, \mu) - \xi$ for

$$\mu_\gamma(\boldsymbol{\delta}|\boldsymbol{x}, y) \propto \Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) - \gamma(\boldsymbol{x}, y)\Big]_+$$

From optimization to sampling

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$
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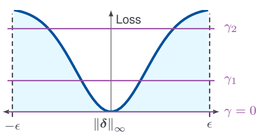
Proposition

For any approximation error, $\exists\,\gamma(\boldsymbol{x}, y)$ such that

$$\mu_\gamma(\boldsymbol{\delta}|\boldsymbol{x}, y) \propto \Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) - \gamma(\boldsymbol{x}, y)\Big]_+$$



[Robey et al., NeurIPS'21]

From optimization to sampling

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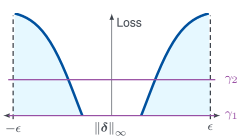

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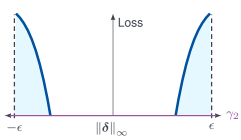
$$\min_{\boldsymbol{\theta}} \ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]$$
$$\Big\updownarrow \approx$$
$$\min_{\boldsymbol{\theta}} \ \sup_{\mu\in\mathcal{P}^2} \ \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\boldsymbol{\delta}\sim\mu_\gamma(\cdot|\boldsymbol{x}_n,y_n)}\left[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}), y_n\big)\right]}_{L(\boldsymbol{\theta},\mu)}$$

Proposition

For any approximation error, $\exists\,\gamma(\boldsymbol{x}, y)$ such that

$$\mu_\gamma(\boldsymbol{\delta}|\boldsymbol{x}, y) \propto \Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}), y\big) - \gamma(\boldsymbol{x}, y)\Big]_+$$



[Robey et al., NeurIPS'21]

From optimization to sampling

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\delta\in\Delta} \text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]$$

$=$

$$\min_{\theta}\ \sup_{\mu\in\mathcal{P}^2}\ \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\delta\sim\mu_{\gamma}(\cdot|x_n,y_n)}\left[\text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]}_{L(\theta,\mu)}$$

Proposition
For any approximation error, $\exists\,\gamma(x,y)$ such that

$$\mu_{\gamma}(\delta|x,y) \propto \left[\text{Loss}\big(f_{\theta}(x+\delta), y\big) - \gamma(x,y)\right]_+$$

From optimization to sampling

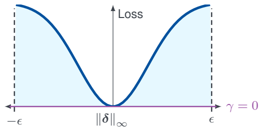
$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\delta\in\Delta} \text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]$$

$\approx$

$$\min_{\theta}\ \sup_{\mu\in\mathcal{P}^2}\ \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\delta\sim\mu_{\gamma}(\cdot|x_n,y_n)}\left[\text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]}_{L(\theta,\mu)}$$

Proposition
For any approximation error, $\exists\,\gamma(x,y)$ such that

$$\mu_{0}(\delta|x,y) \propto \text{Loss}\big(f_{\theta}(x+\delta), y\big)$$



Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\ \min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_{\theta}(x_n), y_n\big) + \lambda\left[\max_{\delta\in\Delta}\text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]$$

- ✓ Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- ✗ Computing the worst-case perturbations
  - gradient ascent → non-convex, underparametrized

Constrained learning for robustness

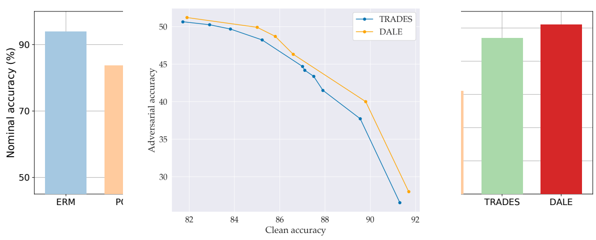
Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\ \min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_{\theta}(x_n), y_n\big) + \lambda\left[\mathbb{E}_{\delta\sim\mu_0(\cdot|x_n,y_n)}\text{Loss}\big(f_{\theta}(x_n+\delta), y_n\big)\right]$$

- ✓ Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- ✓ Computing the worst-case perturbations
  - ~~gradient ascent → non-convex, underparametrized~~ ⇒ sampling

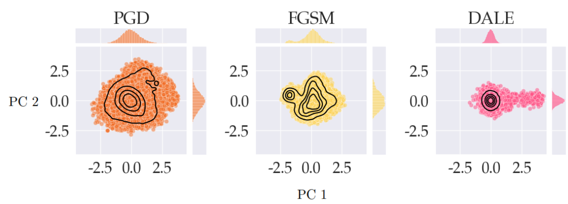
Dual Adversarial LEarning

Problem
Learn an image classifier th



Dual Adversarial LEarning

Problem
Learn an image classifier that is robust to input perturbations



Dual Adversarial LEarning

for $n = 1, \dots, N$:
 $\delta_n \sim$ Random(Δ)
 for $k = 1, \dots, K$:
  $\zeta \sim$ Laplace$(0, I)$
  $\delta_n \leftarrow \text{proj}_{\Delta}\left[\delta_n + \eta\,\text{sign}\left[\nabla_{\delta}\log\left(\text{Loss}\big(f_{\theta_t}(x_n+\delta_n), y_n\big)\right)\right] + \sqrt{2\eta T}\zeta\right]$
 end
 $\theta \leftarrow \theta - \eta\nabla_{\theta}\left[\text{Loss}\big(f_{\theta}(x_n), y_n\big) + \lambda\text{Loss}\big(f_{\theta}(x_n+\delta_n), y_n\big)\right]$
end
$\lambda \leftarrow \left[\lambda + \eta\left(\frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_{\theta}(x_n+\delta_n), y_n\big) - c\right)\right]_+$

HMC sampling: $\delta \sim \mu_0(\cdot|x_n, y_n)$

SGD

GA

Dual Adversarial LEarning

for $n = 1, \dots, N$:
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Dual Adversarial LEarning

1: **for** $n = 1, \dots, N$:

2: $\delta_n \sim$ Random(Δ)

3: **for** $k = 1, \dots, K$:

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6: **end**

7: $\theta \leftarrow \theta - \eta\nabla_\theta\left[\text{Loss}\big(f_\theta(x_n), y_n\big) + \lambda\text{Loss}\big(f_\theta(x_n + \delta_n), y_n\big)\right]$ | SGD

8: **end**

9: $\lambda \leftarrow \left[\lambda + \eta\left(\frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(x_n + \delta_n), y_n\big) - c\right)\right]_+$ | GA

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6: **end**

Gaussian
[Lopes et al., arXiv'19]
[Rusak et al., ECCV'20]
Patches
[Zhong et al., AAAI'20]
[Yun et al., ICCV'19]
...

7: $\theta \leftarrow \theta - \eta\nabla_\theta\left[\text{Loss}\big(f_\theta(x_n), y_n\big) + \lambda\text{Loss}\big(f_\theta(x_n + \delta_n), y_n\big)\right]$ | SGD

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6: **end**

$T \to 0$: "PGD"
[Szegedy et al., ICLR'14]
[Goodfellow et al., ICLR'15]
[Madry et al., ICLR'18]

7: $\theta \leftarrow \theta - \eta\nabla_\theta\left[\text{Loss}\big(f_\theta(x_n), y_n\big) + \lambda\text{Loss}\big(f_\theta(x_n + \delta_n), y_n\big)\right]$ | SGD

8: **end**

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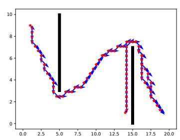
(Manifold) smoothness

Problem
Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

- Labeled data ({State, Action})



Dataset

(Manifold) smoothness

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Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

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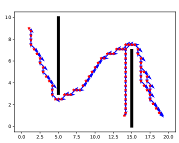
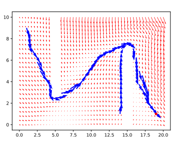


Dataset

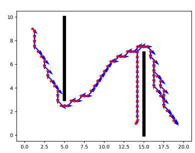


ERM
(85% success)

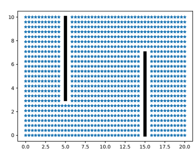
(Manifold) smoothness

Problem
Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

- Labeled data ({State, Action}) and unlabeled data ({State in free space})



Dataset



Dataset

(Manifold) smoothness

Problem

Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

- Labeled data ({State, Action}) and unlabeled data ({State in free space})
- Use {State in free space} to estimate a data manifold $\mathcal{M}$

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \|f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) - \boldsymbol{u}_n\|^2$$
$$\text{subject to} \quad \max_{\boldsymbol{x}} \|\nabla_{\mathcal{M}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\|^2 \leq L$$

[Cerviño et al., ICML'23]

(Manifold) smoothness

Problem

Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

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$$\text{subject to} \quad \max_{\boldsymbol{x}} \|\nabla_{\mathcal{M}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\|^2 \leq L$$

$\mathbb{E}_{\boldsymbol{x}\sim\mu_0}$

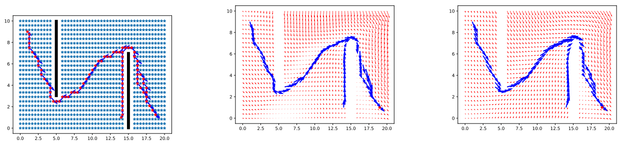
[Cerviño et al., ICML'23]

(Manifold) smoothness

Problem

Learn a smooth (Lipschitz on a manifold) controller that imitates a behavior from limited trajectories

- Labeled data ({Position, Action}) and unlabeled data ({Position})



Dataset

ERM (85% success)

Manifold smoothness (94% success)

[Cerviño et al., ICML'23]

Agenda

Adversarially robust learning

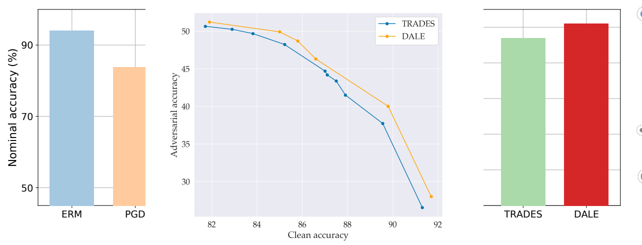
Semi-infinite learning

Probabilistic robustness

Constrained learning for robustness

Problem

Learn an image classifier th



[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Constrained learning for robustness

Problem

Learn an image classifier that is robust to input perturbations

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$
$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \leq \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right] \leq c$$

[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

"Softer" robustness

- Softmax or *log-sum-exp* [Li et al., ICLR'21]

$$\min_{\theta} \mathbb{E}_{(x,y)} \left[\frac{1}{\tau}\log\left(\mathbb{E}_{\delta\sim\mathfrak{m}}\left[e^{\tau\cdot\text{Loss}\left(f_{\theta}(x+\delta),y\right)}\right]\right)\right]$$

  - $\tau \to 0$: classical learning (with randomized data augmentation)
  - $\tau \to \infty$: adversarial robustness (ess sup)

- L_p norms [Rice et al., NeurIPS'21]

$$\min_{\theta} \mathbb{E}_{(x,y)} \left[\mathbb{E}_{\delta\sim\mathfrak{m}}\left[\left|\text{Loss}\big(f_{\theta}(x+\delta),y\big)\right|^{\tau}\right]^{1/\tau}\right]$$

  - $\tau = 1$: classical learning (with randomized data augmentation)
  - $\tau \to \infty$: adversarial robustness (ess sup)

"Softer" robustness

- Softmax or *log-sum-exp* [Li et al., ICLR'21]

$$\min_{\theta} \mathbb{E}_{(x,y)} \left[\frac{1}{\tau}\log\left(\mathbb{E}_{\delta\sim\mathfrak{m}}\left[e^{\tau\cdot\text{Loss}\left(f_{\theta}(x+\delta),y\right)}\right]\right)\right]$$

- L_p norms [Rice et al., NeurIPS'21]

$$\min_{\theta} \mathbb{E}_{(x,y)} \left[\mathbb{E}_{\delta\sim\mathfrak{m}}\left[\left|\text{Loss}\big(f_{\theta}(x+\delta),y\big)\right|^{\tau}\right]^{1/\tau}\right]$$

✖ Computationally challenging (especially as $\tau \to \infty$, i.e., stronger robustness)

✖ No guaranteed advantages (lower sample complexity? improved trade-offs?)

Towards probabilistic robustness

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\left[t(x_n, y_n)\right]$$

subject to
$\text{Loss}\big(f_\theta(x_n + \delta_0), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_1), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_{\sqrt{2}}), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_e), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_\pi), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_4), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_{e^2}), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_{\pi^e}), y_n\big) \le t(x_n, y_n)$
$\text{Loss}\big(f_\theta(x_n + \delta_{e^\pi}), y_n\big) \le t(x_n, y_n)$
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~~$\text{Loss}\big(f_\theta(x_n + \delta_e), y_n\big) \le t(x_n, y_n)$~~
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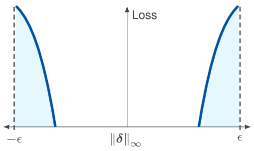
$$\min_{\theta}\ \mathbb{E}_{(x,y)}\left[\ \overset{\approx\, \mathbb{E}_{\delta\sim\mu_0(\cdot|x,y)}}{\cancel{\max_{\delta\in\Delta}}}\ \text{Loss}\big(f_\theta(x+\delta), y\big)\right]$$



[Robey et al., ICML'22 (spotlight)]

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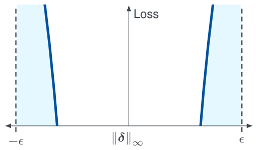
$\text{Loss}\big(f_\theta(x_n + \delta_{25}), y_n\big) \le t(x_n, y_n)$

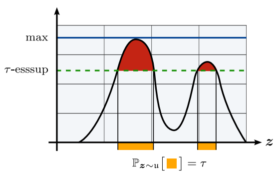
$$\min_{\theta}\ \mathbb{E}_{(x,y)}\left[\ \overset{\approx\, \mathbb{E}_{\delta\sim\mu_0(\cdot|x,y)}}{\cancel{\max_{\delta\in\Delta}}}\ \text{Loss}\big(f_\theta(x+\delta), y\big)\right]$$



[Robey et al., ICML'22 (spotlight)]

Probabilistic robustness

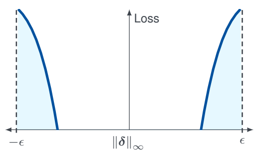
- Probabilistic robustness $\quad \min_{\theta}\ \mathbb{E}_{(x,y)}\left[\tau\text{-esssup}_{\delta\in\Delta}\ \text{Loss}\big(f_\theta(x+\delta), y\big)\right]$
  - $\tau = 1/2$: classical learning (for symmetric m)
  - $\tau = 0$: adversarial robustness (ess sup)

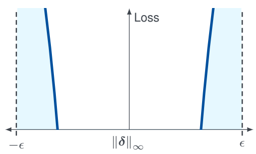


[Robey et al., ICML'22 (spotlight)]

Probabilistic robustness

$$\min_{\theta}\ \mathbb{E}_{(x,y)}\left[\ \overset{\mathbb{E}_{\delta\sim\mu_0(\cdot|x,y)}}{\cancel{\max_{\delta\in\Delta}}}\ \text{Loss}\big(f_\theta(x+\delta), y\big)\right] \qquad \min_{\theta}\ \mathbb{E}_{(x,y)}\left[\ \overset{\mathbb{E}_{\delta\sim\mu_0(\cdot|x,y)}}{\cancel{\tau\text{-esssup}_{\delta\in\Delta}}}\ \text{Loss}\big(f_\theta(x+\delta), y\big)\right]$$





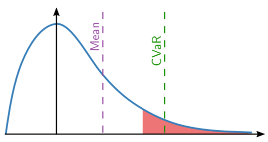
[Robey et al., ICML'22 (spotlight)]

Probabilistic robustness and Risk

- Conditional value at risk:

$$\text{CVaR}_\rho(f) = \mathbb{E}_z\left[f(z) \mid f(z) \ge F_z^{-1}(\rho)\right]$$
$$= \inf_{\alpha\in\mathbb{R}}\ \alpha + \frac{\mathbb{E}_z\left[[f(z)-\alpha]_+\right]}{1-\rho}$$

  - $\text{CVaR}_0(f) = \mathbb{E}_z[f(z)]$
  - $\text{CVaR}_1(f) = \text{ess}\sup_z f(z)$



Proposition
CVaR is the tightest *convex* upper bound of τ-esssup, i.e.,
$\tau\text{-esssup}_z\ f(z) \le \text{CVaR}_{1-\tau}(f)$ with equality when $\rho = 0$ or $\rho = 1$.

[Shapiro et al. Lectures on Stochastic Programming, 2014; Kalogerias et al., IEEE ICASSP'20]

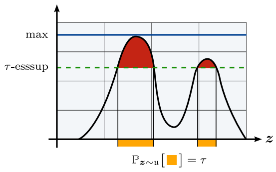
Probabilistically robust learning

for $n = 1, \dots, N$:
$\quad \alpha_0 = 0$
$\quad$ **for** $t = 1, \dots, T$: $\qquad$ SGD (CVaR)
$\quad\quad \delta_t \sim \text{Random}(\Delta)$
$\quad\quad \alpha \leftarrow \alpha - \frac{\eta}{\tau}\Big(\tau - \mathbb{I}\left[\text{Loss}\big(f_\theta(x_n+\delta_t), y_n\big) \ge \alpha\right]\Big)$
$\quad$ **end**
$\quad \theta \leftarrow \theta - \eta\nabla_\theta \underbrace{\left[\text{Loss}\big(f_\theta(x_n+\delta_T), y_n\big) - \alpha\right]_+}_{\approx \text{CVaR}_{1-\tau}\left[\text{Loss}(f_\theta(x_n+\delta), y_n)\right]}$ $\qquad$ SGD (θ)
end

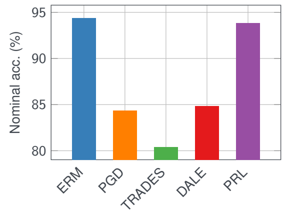
[Robey et al., ICML'22 (spotlight)]

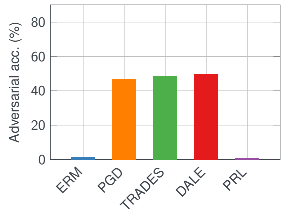
Probabilistic robustness

- Probabilistic robustness $$\min_{\theta} \mathbb{E}_{(x,y)}\left[\tau\text{-}\underset{\delta\in\Delta}{\text{esssup}}\ \text{Loss}\big(f_{\theta}(x+\delta), y\big)\right]$$
  - $\tau = 1/2$: classical learning (for symmetric m)
  - $\tau = 0$: adversarial robustness (ess sup)
  - Potentially better sample complexity
    [Robey et al., ICML'22 (spotlight)]
    [Raman et al., NeurIPS ML Safety Workshop'22]
  - Better performance trade-off
    [Robey et al., ICML'22 (spotlight)]

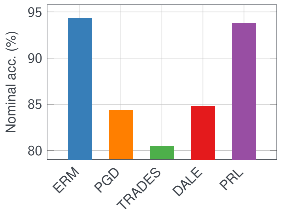


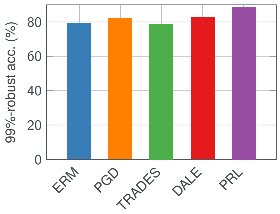
Probabilistically robust learning





Probabilistically robust learning





Summary

- **Semi-infinite constrained learning is the a tool to enforce worst-case requirements**
- **Semi-infinite constrained learning...**
- **...but possible. How?**

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  e.g., robustness [Robey et al., NeurIPS'21], invariance [Hounie et al., ICML'23], smoothness [Cerviño et al., ICML'23]...
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- **Semi-infinite constrained learning...**
  Learning problem with an infinite number of constraints
- **...but possible. How?**
  Using a hybrid sampling–optimization algorithm or, in the case of probabilistic robustness, a *tight* convex relaxation (CVaR) [Robey et al., ICML'22]

Break